Uniform Continuity and Brézis-Lieb Type Splitting for Superposition Operators in Sobolev Space

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Using concentration-compactness arguments we prove a variant of the Brézis-Lieb-Lemma under weaker assumptions on the nonlinearity than known before. An intermediate result on the uniform continuity of superposition operators in Sobolev space is of independent interest.

1 Introduction

In their seminal paper [6] Brézis and Lieb prove a result about the decoupling of certain integral expressions, which has been used extensively in

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the calculus of variations. Using concentration compactness arguments in the spirit of Lions [14–16] we prove a variant of this lemma under weaker assumptions on the nonlinearity than known before. To describe a special case of the Brézis-Lieb lemma, suppose that Ω is an unbounded domain in $\mathbb{R}^N$, $p > 1$, $f(t) := |t|^p$ for $t \in \mathbb{R}$, and (u_n) a bounded sequence in $L^p(\Omega)$ that converges pointwise almost everywhere to some function u. If one denotes by $\mathcal{F}\colon L^p(\Omega) \to L^1(\Omega)$ the superposition operator induced by f, i.e., $\mathcal{F}(v)(x) := f(v(x))$, then the result in [6] implies that $u \in L^p(\Omega)$ and

$$\mathcal{F}(u_n) - \mathcal{F}(u_n - u) \to \mathcal{F}(u) \qquad \text{in } L^1(\Omega), \text{ as } n \to \infty. \tag{1.1}$$

The same conclusion is obtained in that paper for more general functions, imposing conditions that are satisfied for continuous convex f with $f(0) = 0$, and imposing additional conditions on the sequence (u_n).

A different approach to the decoupling of superposition operators along sequences of functions rests on certain regularity assumptions on f. For example, assume that $f \in C^1(\mathbb{R})$ satisfies

$$\sup_{t \in \mathbb{R}} \frac{|f'(t)|}{|t|^{p-1}} < \infty. \tag{1.2}$$

Then the proof of [19, Lemma 8.1] can easily be extended to obtain (1.1). See also the slightly more general [7, Lemma 1.3], where f is allowed to depend on x explicitly.

Our aim is to give a decoupling result under a different set of hypotheses that applies to a much larger class of functions f than considered above, within a certain range of exponents p. In particular, we do not impose any convexity type assumptions on f as was done in [6], nor any regularity assumptions as in [7,19] apart from continuity. The price we pay for relaxing the hypotheses on f is that we need to restrict the range of allowed growth exponents p in comparison with [6], that we need to assume some type of translation invariance for Ω, and that the decoupling result only applies to a smaller set of admissible sequences, namely sequences that converge weakly in $H^1(\Omega)$. Nevertheless, the numerous applications in the Calculus of Variations for PDEs where these extra assumptions are satisfied justify the new set of hypotheses.

To keep the presentation simple and highlight the main idea, we only treat the case $\Omega = \mathbb{R}^N$. From here on, function spaces are taken over $\mathbb{R}^N$ unless otherwise noted. It would be possible to consider other domains or superposition operators between other spaces, and we plan to do so in forthcoming work. Nevertheless, we do allow a periodic dependency of f on the space variable.

To explain our results we formalize the notion of decoupling:

Definition 1.1. Suppose that X and Y are Banach spaces. Consider a map $\mathcal{F}\colon X \to Y$, a sequence $(u_n) \subseteq X$ and $u \in X$. We say that $\mathcal{F}$ *BL-splits along* (u_n) *with respect to* u (BL being an abbreviation for Brézis-Lieb) if

$$\|\mathcal{F}(u_n) - \mathcal{F}(u_n - u) - \mathcal{F}(u)\|_Y \to 0.$$

We say that $\mathcal{F}$ *almost BL-splits along* (u_n) *with respect to* u if, starting with any subsequence of (u_n), we can pass to a subsequence such that there is a sequence $(v_n) \subseteq X$ such that $\|v_n - u\|_X \to 0$ and

$$\|\mathcal{F}(u_n) - \mathcal{F}(u_n - v_n) - \mathcal{F}(u)\|_Y \to 0.$$

If u is a limit of (u_n) in some unambiguous sense then we frequently omit to mention that (almost) BL-splitting is *with respect to* u.

By [6], the map $f(u) = |u|^p$ induces a map $\mathcal{F}\colon L^p \to L^1$ that BL-splits along pointwise a.e. converging bounded sequences in L^p with respect to their pointwise a.e. limits. On the other hand, the technique used to prove [1, Lemma 3.2] (and the related results in [10, 11]) yields the following: if $f \in C(\mathbb{R})$ satisfies

$$\sup_{t\in\mathbb{R}} \frac{|f(t)|}{|t|^p} < \infty \tag{1.3}$$

then the induced superposition operator $\mathcal{F}\colon L^p \to L^1$ almost BL-splits along any L^p_{loc}-converging bounded sequence in L^p with respect to its limit in L^p_{loc}, see Theorem 2.1(a) below. This result is basically Lions' approach, with a simplifying twist. If in addition $\mathcal{F}$ is uniformly continuous on bounded subsets of L^p then it is easy to see that it BL-splits along any L^p_{loc}-converging bounded sequence in L^p with respect to its limit in L^p_{loc}, see [2, Lemma 6.3]. For example, this holds true if (1.2) is satisfied.

We illustrate the distinction between BL-splitting and almost BL-splitting by the following examples:

Example 1.2. If $p > 1$ and if either $f(t) := \cos(\pi t)|t|^p$ or $f(t) := \cos(\pi/t)|t|^p$ then there is a bounded sequence (u_n) in L^p that converges in L^p_{loc} and pointwise a.e. to a function u such that the induced continuous superposition operator $\mathcal{F} := L^p \to L^1$ does not BL-split along any subsequence of (u_n) with respect to u. On the other hand, $\mathcal{F}$ almost BL-splits along any L^p_{loc}-converging bounded sequence in L^p with respect to its limit in L^p_{loc}. Hence $\mathcal{F}$ is not uniformly continuous on bounded subsets of L^p and neither the general conditions used in [6] nor (1.2) are satisfied for f in these examples.

The sequences mentioned in the example are provided in Section 4 below.

Our main interest is to avoid condition (1.2), or any other conditions on f that ensure uniform continuity on bounded subsets of L^p (e.g., a local Hölder condition, together with an appropriate growth bound on the Hölder constants on bounded intervals). Our result below states that it is sufficient to restrict to bounded subsets of H^1 instead.

In this context we now formulate our main theorem, in a slightly more general setting than what we considered above. A function $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a *Caratheodory function* if f is measurable and if $f(x, \cdot)$ is continuous for almost every $x \in \mathbb{R}^N$. The induced superposition operator on functions $u\colon \mathbb{R}^N \to \mathbb{R}$ is then given by $\mathcal{F}(u)(x) := f(x, u(x))$. If A is a real invertible $N \times N$-matrix then f is said to be A-periodic in its first argument if $f(x + Ak, t) = f(x, t)$ for all $x \in \mathbb{R}^N$, $k \in \mathbb{Z}^N$, and $t \in \mathbb{R}$.

Denote by $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := \infty$ if $N = 1$ or $N = 2$ the critical Sobolev exponent for H^1. Recall the continuous and compact embedding of the Sobolev space $H^1(U)$ in $L^p(U)$ for $p \in [2, 2^*)$ if $U \subseteq \mathbb{R}^N$ is a bounded domain.

Theorem 1.3. *Consider $\mu > 0$, $\nu \geq 1$, and $C_0 > 0$, such that $p := \mu\nu \in (2, 2^*)$. Suppose that $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function that satisfies*

$$|f(x,t)| \leq C_0 |t|^\mu \qquad \textit{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{1.4}$$

and which is A-periodic in its first argument, for some invertible matrix $A \in \mathbb{R}^{N\times N}$. Denote by $\mathcal{F}\colon L^p \to L^\nu$ the continuous superposition operator induced by f. Then $\mathcal{F}$ is uniformly continuous on bounded subsets of H^1 with respect to the L^p-L^ν-norms and hence also with respect to the H^1-L^ν-norms. Moreover, $\mathcal{F}\colon H^1 \to L^\nu$ BL-splits along weakly convergent sequences in H^1 with respect to their weak limit.

Our proof of Theorem 1.3 has similarities with the proof of [18, Theorem 3.1] but involves an intermediate cut-off step in the proof of Theorem 2.1. Essentially, we first prove almost BL-splitting of $\mathcal{F}$ along weakly converging sequences in H^1 with respect to their weak limit, using the concentration function and the compactness of the Sobolev embedding $H^1(U) \hookrightarrow L^p(U)$, for $p \in [2, 2^*)$ and for a bounded domain U. Then we collect the possible mass loss at infinity along subsequences with the help of Lions' Vanishing Lemma, employing the assumption $p > 2$.

Remark 1.4. Theorem 1.3 applies in particular to the functions considered in Example 1.2 when $\nu = 1$ and $\mu = p \in (2, 2^*)$. On the other hand, for $f(t) := \cos(\pi/t)t^2$ there is a sequence in H^1 that converges weakly but that possesses no subsequence along which $f\colon H^1 \to L^1$ BL-splits with respect to the weak limit. The same is true for $f(t) := \cos(\pi t)t^{2^*}$. In this sense, Theorem 1.3 is optimal, that is, it cannot be extended in this generality to include the cases $p = \mu\nu = 2$ and $p = \mu\nu = 2^*$. The existence of these counterexamples is proved in Section 4.

Of course, by Sobolev's embedding theorem, a map $L^p \to L^\nu$ that BL-splits along L^p_{loc}-converging bounded sequences in L^p with respect to their limits in L^p_{loc} also BL-splits in H^1 along weakly convergent sequences with respect to their weak limits. Therefore Theorem 2.1(a), together with (1.2) (or the weaker Hölder condition with growth bound), yields BL-splitting maps along weakly convergent sequences in H^1 with respect to weak limits even for $p = 2$ and $p = 2^*$.

Remark 1.5. The result also holds true in a slightly restricted sense for functions f that are sums of functions as in Theorem 1.3, i.e., functions that satisfy merely

$$|f(x,t)| \le C_0(|t|^{\mu_1} + |t|^{\mu_2}) \qquad \text{for all } x \in \mathbb{R}^N,\ t \in \mathbb{R},$$

where $\mu_i\nu \in (2, 2^*)$ for $i = 1, 2$. In that case, $\mathcal{F}\colon H^1 \to L^\nu$ is uniformly continuous on bounded subsets of H^1 with respect to the H^1-L^ν norms, and $\mathcal{F}$ BL-splits along weakly convergent sequences in H^1 with respect to their weak limits.

Remark 1.6. The uniform continuity of operators $\mathcal{F}$ on bounded subsets of H^1 has been used, for example, in the proof of [18, Lemma 3.4]. Nevertheless, we are not aware of a published proof of this fact, which is nontrivial in the generality stated in Theorem 1.3. Note that the uniform continuity of $\mathcal{F}\colon H^1(U) \to L^\nu(U)$ on bounded subsets of $H^1(U)$ is trivial if U is bounded, by the compact Sobolev embedding $H^1(U) \subseteq L^p(U)$.

We now discuss additional aspects and applications of the results presented above. To this end we return to a simple setting on $\mathbb{R}^N$. Suppose that $f \in C(\mathbb{R})$ satisfies (1.3) with $p \in (2, 2^*)$ and consider the functional $\Phi\colon H^1 \to \mathbb{R}$ given by

$$\Phi(u) := \int_{\mathbb{R}^N} f(u).$$

To prove the existence of a minimizer in typical variational problems involving Φ, Lions [14,15] introduces the concentration-compactness principle. It is a tool to exclude the possibility of vanishing and of dichotomy along a minimizing sequence (u_n), in order to obtain compactness of the sequence. Here we are only concerned with dichotomy. In this case, the sequence (u_n) is approximated by $(u_n^1 + u_n^2)$, where $\operatorname{dist}(\operatorname{supp}(u_n^1), \operatorname{supp}(u_n^2)) \to \infty$. For *local* functionals like Φ it then follows easily that $\Phi(u_n)$ is approximated by $\Phi(u_n^1) + \Phi(u_n^2)$, a fact that yields, together with a hypothesis about energy levels, a contradiction. Clearly, the same can be achieved if Φ BL-splits along (u_n) in a suitable way. Before our Theorem 1.3, Lions' approach to concentration compactness was more general, in that, besides continuity and appropriate growth bounds, no extra regularity hypotheses need to be placed on f. On the other hand, the arguments are more involved than when using BL-splitting because one has to insert cut-off functions to obtain sequences u_n^1 and u_n^2 with disjoint supports. As a consequence, it is difficult to give a purely functional (abstract) presentation of Lions' approach.

To explain the advantage of an abstract presentation using BL-splitting, we note that to treat nonlocal functionals of convolution type, e.g.,

$$\Psi(u) := \int_{\mathbb{R}^N} (f * h(u))h(u),$$

the property of disjoint supports is not as effective anymore. In the convolution, the supports get "smeared out" and one has to control the interaction with more involved estimates, see page 123 of [14]. This is aggravated when one also has to consider the decoupling of *derivatives* of Ψ. We have shown in [2] that using BL-splitting is effective in situations involving nonlocal functionals. Moreover, BL-splitting even survives certain nonlocal operations, like the saddle point reduction, see [2, Theorem 5.1].

For particular cases there are other approaches to avoid conditions on f besides continuity and growth bounds. We reformulate and simplify the following cited results slightly to adapt them to our setting and notation. In [3] we proved, for $f \in C(\mathbb{R})$ satisfying (1.4) with $\mu := p-1$, and setting $\nu := p/(p-1)$, that the map $\Gamma\colon H^1 \to H^{-1}$, given by

$$\Gamma(u)v := \int_{\mathbb{R}^N} f(u)v,$$

BL-splits along a weakly convergent sequence if the weak limit is a function tending to 0 as $|x| \to \infty$. Another result was given in [13, Lemma 7.2], when $f \in C(\mathbb{R})$ satisfies (1.4) with $\mu := p-2$ and $\nu := p/(p-2)$: The map $\Lambda\colon H^1 \to \mathcal{L}_2(H^1, \mathbb{R})$ (here $\mathcal{L}_2(H^1, \mathbb{R})$ denotes the space of bounded bilinear maps from H^1 into $\mathbb{R}$), given by

$$\Lambda(u)[v,w] := \int_{\mathbb{R}^N} f(u)vw,$$

is uniformly continuous on bounded subsets of H^1. Together with the almost BL-splitting of Λ given by Theorem 2.1 below this yields BL-splitting for Λ along weakly convergent sequences. Note that the idea of the proof of the latter result does not apply for the maps Φ and Γ defined above (under the respective growth bounds on f). In both cases our result here is stronger, since we show uniform continuity and BL-splitting into the spaces L^ν, which are continuously embedded in H^{-1} and $\mathcal{L}_2(H^1, \mathbb{R})$, respectively.

A different application of Theorem 1.3, that is independent of variational methods, is the general study of maps that are uniformly continuous on a subset of an infinite dimensional Hilbert space. These play a role in infinite dimensional potential theory [5, 12] or, more generally, in the theory of stochastic equations in infinite dimensions [8, 9, 17].

The paper is structured as follows. In Section 2 we treat almost BL-splitting of $\mathcal{F}$ along bounded sequences in L^p that converge in L^p_{loc}, and along weakly convergent sequences in H^1. In Section 3 we prove the uniform continuity of $\mathcal{F}$ on bounded subsets of H^1 and BL-splitting of $\mathcal{F}$ along weakly convergent sequences. In Section 4 we prove the claims made in Example 1.2.

2 Almost BL-Splitting

In this section we prove a result on the almost BL-splitting of superposition operators in L^p along bounded sequences that converge in L^p_{loc}, and in H^1 along weakly convergent sequences. This is a variation on Lions' approach in [14]. Note that here the periodicity assumption in x is not needed.

If $r \in [1, \infty]$ then denote by $|\cdot|_r$ the norm of L^r.

Theorem 2.1. *Consider $\mu > 0$, $\nu \geq 1$, and $C_0 > 0$, such that $p := \mu\nu \geq 1$. Suppose that $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function that satisfies* (1.4)*. Denote by $\mathcal{F}$ the superposition operator on real functions induced by f.*

(a) *If $(u_n) \subseteq L^p$ is bounded and converges in L^p_{loc} to a function u, then $u \in L^p$ and $\mathcal{F}\colon L^p \to L^\nu$ almost BL-splits along (u_n) with respect to u.*

(b) *If $p \in [2, 2^*)$ and $u_n \rightharpoonup u$ in H^1 then $\mathcal{F}\colon H^1 \to L^\nu$ almost BL-splits along (u_n) with respect to u.*

(c) *In* (b)*, if in addition $(\bar{u}_n) \subseteq H^1$ converges weakly and $|u_n - \bar{u}_n|_p \to 0$ as $n \to \infty$ then $\bar{u}_n \rightharpoonup u$ in H^1 and $\mathcal{F}$ almost BL-splits along (u_n) and $(\bar{u}_n)$ with respect to u, preserving subsequences and the auxiliary sequence (v_n) in the following sense: for any subsequence n_k there* *is a subsequence* n_{k_ℓ} *and* (v_ℓ) *such that* $v_\ell \to u$ *in* H^1 *and, writing* $u_\ell := u_{n_{k_\ell}}$ *and* $\bar{u}_\ell := \bar{u}_{n_{k_\ell}}$ *we have*

$$\mathcal{F}(u_\ell) - \mathcal{F}(u_\ell - v_\ell) \to \mathcal{F}(u)$$

and

$$\mathcal{F}(\bar{u}_\ell) - \mathcal{F}(\bar{u}_\ell - v_\ell) \to \mathcal{F}(u).$$

For the proof, let B_R denote, for $R > 0$, the open ball in $\mathbb{R}^N$ with center 0 and radius R.

Proof. **(a):** From (1.4) and from the theory of superposition operators [4] it follows that $\mathcal{F}\colon L^p(U) \to L^\nu(U)$ is continuous for any open subset U of $\mathbb{R}^N$. For $n \in \mathbb{N}$ define $Q_n\colon [0,\infty) \to [0,\infty)$ by

$$Q_n(R) := \int_{B_R} |u_n|^p.$$

The functions Q_n are uniformly bounded and nondecreasing. We may assume that (Q_n) converges pointwise almost everywhere to a bounded nondecreasing function Q [14]. It is easy to build a sequence $R_n \to \infty$ such that for every $\varepsilon > 0$ there is $R > 0$, arbitrarily large, with

$$\limsup_{n\to\infty}(Q_n(R_n) - Q_n(R)) \le \varepsilon.$$

Hence

$$\forall \varepsilon > 0\ \exists R > 0\colon \limsup_{n\to\infty} \int_{B_{R_n}\setminus B_R} |u_n|^p \le \varepsilon \qquad \text{and} \qquad \int_{\mathbb{R}^N\setminus B_R} |u|^p \le \varepsilon. \tag{2.1}$$

Consider a smooth cut off function $\eta\colon [0,\infty) \to [0,1]$ such that $\eta \equiv 1$ on $[0,1]$ and $\eta \equiv 0$ on $[2,\infty)$. Set $v_n(x) := \eta(2|x|/R_n)u(x)$. Then

$$\lim_{n\to\infty} v_n = u \qquad \text{in } L^p. \tag{2.2}$$

From the continuity of $\mathcal{F}$ on $L^p(B_R)$, $v_n = u$ on B_R, $\lim_{n\to\infty} u_n = u$ in $L^p(B_R)$, and $f(x,0) = 0$ for a.e. $x \in \mathbb{R}^N$ we obtain

$$\lim_{n\to\infty} \int_{B_R} \Big| f(x,u_n) - f(x,u_n - v_n) - f(x,v_n)\Big|^\nu \,\mathrm{d}x = \lim_{n\to\infty} \int_{B_R} \Big| f(x,u_n) - f(x,u_n - u) - f(x,u)\Big|^\nu \,\mathrm{d}x = 0.$$

Since $v_n \equiv 0$ in $\mathbb{R}^N \backslash B_{R_n}$, this in turn yields for any $\varepsilon > 0$ and R chosen accordingly, as in (2.1),

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^N} &|f(x,u_n) - f(x,u_n - v_n) - f(x,v_n)|^\nu \,\mathrm{d}x \\ &= \limsup_{n\to\infty} \int_{B_{R_n}\backslash B_R} |f(x,u_n) - f(x,u_n - v_n) - f(x,v_n)|^\nu \,\mathrm{d}x \\ &\leq C \limsup_{n\to\infty} \int_{B_{R_n}\backslash B_R} (|u_n|^\mu + |u_n - v_n|^\mu + |v_n|^\mu)^\nu \\ &\leq C \limsup_{n\to\infty} \int_{B_{R_n}\backslash B_R} (|u_n|^p + |u|^p) \\ &\leq C\varepsilon,\end{aligned}$$

where C is independent of ε. Letting ε tend to 0 and using (2.2) we obtain

$$\lim_{n\to\infty} \Big|\mathcal{F}(u_n) - \mathcal{F}(u_n - v_n) - \mathcal{F}(u)\Big|_\nu = 0.$$

(b): The continuous embedding $H^1 \hookrightarrow L^p$ implies that (u_n) is bounded in L^p, and the compact embedding $H^1(U) \hookrightarrow L^p(U)$ for bounded U implies that $u_n \to u$ in L^p_{loc}. Defining v_n as in (a) we therefore obtain that

$$v_n \to u \qquad \text{in } H^1,$$

and $\mathcal{F}$ almost BL-splits along (u_n) with respect to u by (a).

(c): Since $|u_n - \bar{u}_n|_p \to 0$ and $u_n \to v$ in L^p_{loc} it follows that $\bar{u}_n \rightharpoonup v$ in H^1. Taking R large enough, (2.1) also holds true if we replace u_n by $\bar{u}_n$. Therefore, after passing to a subsequence for (u_n), and using the same subsequence for $(\bar{u}_n)$, we obtain

$$\lim_{n\to\infty} \Big|\mathcal{F}(\bar{u}_n) - \mathcal{F}(\bar{u}_n - v_n) - \mathcal{F}(u)\Big|_\nu = 0. \qquad \square$$

3 Uniform Continuity

Here we prove uniform continuity on bounded subsets of H^1, making use of the periodicity of f in x. As a consequence, we also obtain BL-splitting along weakly convergent sequences in H^1.

For simplicity we will only prove the case $A = I$ (the identity transformation). The general case follows in an analogous manner. Denote the respective translation action of the additive group $\mathbb{Z}^N$ on functions $u\colon \mathbb{R}^N \to \mathbb{R}$ by

$$(a \star u)(x) := u(x-a), \qquad a \in \mathbb{Z}^N,\ x \in \mathbb{R}^N.$$

Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product in H^1, defined by

$$\langle u, v \rangle := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv),$$

and let $\|\cdot\|$ denote the associated norm. Also denote by w-lim the weak limit of a weakly convergent sequence.

We first recall a functional consequence of Lions' Vanishing Lemma, [15, Lemma I.1.].

Lemma 3.1. *Suppose for a sequence $(u_n) \subseteq H^1$ that $a_n \star u_n \rightharpoonup 0$ in H^1 for every sequence $(a_n) \subseteq \mathbb{Z}^N$. Then $u_n \to 0$ in L^p for all $p \in (2, 2^*)$.*

Proof. Note first that (u_n) is bounded in H^1 since $u_n \rightharpoonup 0$ in H^1. We claim that

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_1} |u_n|^2 \to 0 \qquad \text{as } n \to \infty. \tag{3.1}$$

If the claim were not true there would exist $\varepsilon > 0$ and a sequence $(y_n) \subseteq \mathbb{R}^N$ such that, after passing to a subsequence of (u_n),

$$\int_{y_n+B_1} |u_n|^2 \geq \varepsilon.$$

Pick $(a_n) \subseteq \mathbb{Z}^N$ such that $|a_n + y_n|_\infty < 1$ for all n. With $R := \sqrt{N} + 1$ it follows that $a_n + y_n + B_1 \subseteq B_R$ and hence

$$\int_{B_R} |a_n \star u_n|^2 \geq \varepsilon$$

for all n. We reach a contradiction since $a_n \star u_n \rightharpoonup 0$ in H^1 and hence $a_n \star u_n \to 0$ in $L^2(B_R)$ by the theorem of Rellich and Kondrakov. Therefore (3.1) holds true.

The claim of the theorem now follows from [15, Lemma I.1.] with $p = q = 2$. Compare also with [18, Lemma 3.3]. □

Proof of Theorem 1.3. We start by proving the uniform continuity. Let $(u^0_{i,n})_{n\in\mathbb{N}_0}$ be bounded sequences in H^1 for $i = 1,2$ and set $C_1 := \max_{i=1,2}\limsup_{n\to\infty}\|u^0_{i,n}\|$. Suppose for a contradiction that

$$|u^0_{1,n} - u^0_{2,n}|_p \to 0 \qquad \text{as } n\to\infty, \tag{3.2}$$

and that there is $C_2 > 0$ such that

$$|\mathcal{F}(u^0_{1,n}) - \mathcal{F}(u^0_{2,n})|_\nu \geq C_2 \qquad \text{for all } n. \tag{3.3}$$

Successively we will define infinitely many sequences $(a^k_n)_n \subseteq \mathbb{Z}^N$ and $(u^k_{i,n})_n \subseteq H^1$, $i = 1,2$, indexed by $k \in \mathbb{N}_0$ and strictly increasing functions $\varphi_k\colon \mathbb{N}\to\mathbb{N}$ with the following properties:

$$\max_{i=1,2}\limsup_{n\to\infty}\|u^k_{i,n}\| \leq C_1, \tag{3.4}$$

$$\lim_{n\to\infty}|u^k_{1,n} - u^k_{2,n}|_p = 0, \tag{3.5}$$

$$\liminf_{n\to\infty}|\mathcal{F}(u^k_{1,n}) - \mathcal{F}(u^k_{2,n})|_\nu \geq C_2, \tag{3.6}$$

$$\operatorname*{w-lim}_{n\to\infty}\left(-a^\ell_{\psi^{k-1}_\ell(n)}\right)\star u^k_{i,n} = 0 \qquad \text{in } H^1, \text{ if } 0\leq \ell < k, \text{ for } i = 1,2, \tag{3.7}$$

and

$$\lim_{n\to\infty}|a^m_{\psi^\ell_m(n)} - a^\ell_n| = \infty \quad \text{if } 0 \leq m < \ell < k. \tag{3.8}$$

Here

$$\begin{aligned} \psi^k_\ell &:= \varphi_{\ell+1}\circ\varphi_{\ell+2}\circ\cdots\circ\varphi_k \qquad && \text{if } \ell = -1,0,1,\ldots,k-1 \\ \psi^k_k &:= \mathrm{id}_{\mathbb{N}}. \end{aligned}$$

We need to say something about the extraction of subsequences. In order to obtain φ_k, $(a^k_n)_n$, and $(u^{k+1}_{i,n})_n$ from $(u^k_{i,n})$, we first pass to a subsequence $(u^k_{i,\varphi_k(n)})_n$ of $(u^k_{i,n})_n$ and then use its terms in the construction. Once the new sequences $(a^k_n)_n$ and $(u^{k+1}_{i,n})_n$ are built we may remove a finite number of terms at their start, modifying φ_k accordingly, with the goal of obtaining additional properties. Beginning with the following iteration there are no more retrospective changes to the sequences already built. This is to assure a well defined infinite sequence of sequences, from which eventually we take the diagonal sequence. In this setting it seems clearer to make the selection of subsequences explicit, contrary to what is usually done when using concentration compactness methods [14–16] or when proving a variational splitting lemma.

For $k = 0$ the properties (3.4)– (3.8) are fulfilled by the definition of C_1 and by (3.2) and (3.3). Assume now that (3.4)–(3.8) hold for some $k \in \mathbb{N}_0$. Denote by W_k the set of $v \in H^1$ such that there are a sequence $(a_n) \subseteq \mathbb{Z}^N$ and a subsequence of $(u^k_{1,n})$ with $\text{w-lim}_{n\to\infty} a_n \star u^k_{1,n} = v$ in H^1.

If $\text{w-lim}_{n\to\infty} a_n \star u^k_{1,n} = 0$ in H^1 were true for all sequences $(a_n) \subseteq \mathbb{Z}^N$, by Lemma 3.1 it would follow that $\lim_{n\to\infty} u^k_{1,n} = 0$ in L^p. Equation (3.5) and the continuity of $\mathcal{F}$ on L^p would lead to a contradiction with (3.6). Therefore

$$q_k := \sup_{v\in W_k} \|v\| \in (0, C_1].$$

Pick $v^k \in W_k$ such that

$$\|v^k\| \geq \frac{q_k}{2} > 0. \tag{3.9}$$

There are $(a^k_n)_n \subseteq \mathbb{Z}^N$ and a strictly increasing function $\varphi_k \colon \mathbb{N} \to \mathbb{N}$ such that $\text{w-lim}_{n\to\infty}(-a^k_n) \star u^k_{1,\varphi_k(n)} = v^k$ in H^1. By (3.5) and by Theorem 2.1(b) and (c) there exists a sequence $(v^k_n)_n \subseteq H^1$ such that

$$\lim_{n\to\infty} v^k_n = v^k, \qquad \text{in } H^1, \tag{3.10}$$

$$\operatorname*{w-lim}_{n\to\infty}(-a^k_n) \star u^k_{i,\varphi_k(n)} = v^k, \qquad \text{in } H^1, \text{ for } i = 1, 2, \tag{3.11}$$

and

$$\lim_{n\to\infty}\Big|\mathcal{F}((-a^k_n)\star u^k_{i,\varphi_k(n)}) - \mathcal{F}((-a^k_n)\star u^k_{i,\varphi_k(n)} - v^k_n) - \mathcal{F}(v^k)\Big|_\nu = 0, \qquad i = 1, 2.$$

Set $u^{k+1}_{i,n} := u^k_{i,\varphi_k(n)} - a^k_n \star v^k_n$. By the equivariance of $\mathcal{F}$ and the invariance of the involved norms under the $\mathbb{Z}^N$-action,

$$\lim_{n\to\infty}\Big|\mathcal{F}(u^k_{i,\varphi_k(n)}) - \mathcal{F}(u^{k+1}_{i,n}) - \mathcal{F}(a^k_n \star v^k)\Big|_\nu = 0, \qquad \text{for } i = 1, 2, \tag{3.12}$$

and, since by (3.11) $\|\cdot\|^2$ BL-splits along $(-a_n^k)\star u^k_{i,\varphi_k(n)}$ with respect to v^k,

$$\lim_{n\to\infty}\Big|\|u^k_{i,\varphi_k(n)}\|^2-\|u^{k+1}_{i,n}\|^2-\|v^k\|^2\Big|=0,\qquad\text{for } i=1,2. \tag{3.13}$$

Equations (3.13) and (3.4) (for k) imply that

$$\max_{i=1,2}\limsup_{n\to\infty}\|u^{k+1}_{i,n}\|\le C_1,$$

hence (3.4) for $k+1$. The definition of the sequences $u^{k+1}_{i,n}$ and (3.5) (for k) imply that

$$\lim_{n\to\infty}|u^{k+1}_{1,n}-u^{k+1}_{2,n}|_p=\lim_{n\to\infty}|u^k_{1,\varphi_k(n)}-u^k_{2,\varphi_k(n)}|_p=0, \tag{3.14}$$

hence (3.5) for $k+1$. It follows from (3.12) and (3.6) (for k) that

$$\liminf_{n\to\infty}|\mathcal{F}(u^{k+1}_{1,n})-\mathcal{F}(u^{k+1}_{2,n})|_\nu=\liminf_{n\to\infty}|\mathcal{F}(u^k_{1,\varphi_k(n)})-\mathcal{F}(u^k_{2,\varphi_k(n)})|_\nu\ge C_2, \tag{3.15}$$

hence (3.6) for $k+1$. Last but not least, from (3.7) (for k), (3.9), and (3.11) it follows that

$$\lim_{n\to\infty}|a^m_{\psi^k_m(n)}-a^k_n|=\infty\qquad\text{if } m<k. \tag{3.16}$$

Since (3.8) is true for k, together with (3.16) we obtain (3.8) for $k+1$. Moreover, (3.16), (3.7) (for k) and (3.10) yield

$$\begin{aligned}\operatorname*{w-lim}_{n\to\infty}(-a^\ell_{\psi^k_\ell(n)})\star u^{k+1}_{i,n}&=\operatorname*{w-lim}_{n\to\infty}\Big((-a^\ell_{\psi^{k-1}_\ell(\varphi_k(n))})\star u^k_{i,\varphi_k(n)}-(a^k_n-a^\ell_{\psi^k_\ell(n)})\star v^k_n\Big)\\&=0,\qquad\text{in } H^1,\text{ if } \ell<k.\end{aligned}$$

By the definition of a^k_n,

$$\operatorname*{w-lim}_{n\to\infty}(-a^k_n)\star u^{k+1}_{i,n}=\operatorname*{w-lim}_{n\to\infty}\Big((-a^k_n)\star u^k_{i,\varphi_k(n)}-v^k_n\Big)=0,\qquad\text{in } H^1.$$

This proves (3.7) for $k+1$.

We now skip a finite number of elements of the sequences constructed in this induction step and adapt φ_k accordingly. Choosing $m\in\mathbb{N}$ large enough, by (3.14) and (3.15) we obtain

$$|u^{k+1}_{1,m+n}-u^{k+1}_{2,m+n}|_p\le\frac{1}{k+1}$$

and

$$|\mathcal{F}(u^{k+1}_{1,m+n}) - \mathcal{F}(u^{k+1}_{2,m+n})|_\nu \geq C_2 - \frac{1}{k+1}$$

for all $n \in \mathbb{N}$. Property (3.8) (for $k+1$) implies that

$$\lim_{n\to\infty} |a^m_{\psi^k_m(n)} - a^\ell_{\psi^k_\ell(n)}| = \lim_{n\to\infty} |a^m_{\psi^\ell_m(\psi^k_\ell(n))} - a^\ell_{\psi^k_\ell(n)}| = \infty, \qquad \text{if } m < \ell \leq k.$$

Since $\|\cdot\|^2$ BL-splits along weakly convergent sequences this yields, together with (3.10), that

$$\lim_{n\to\infty} \left\| \sum_{j=\ell}^{k} a^j_{\psi^k_j(n)} \star v^j_{\psi^k_j(n)} \right\|^2 = \sum_{j=\ell}^{k} \|v^j\|^2$$

for all $\ell \leq k$. For large enough m this implies

$$\left\| \sum_{j=\ell}^{k} a^j_{\psi^{k-1}_j(\varphi_k(m+n))} \star v^j_{\psi^{k-1}_j(\varphi_k(m+n))} \right\|^2 \leq 2 \sum_{j=\ell}^{k} \|v^j\|^2, \qquad \text{for all } n \in \mathbb{N} \text{ and } \ell \leq k.$$

Fixing m with these properties, writing $u^{k+1}_{i,n}$, a^k_n, and v^k_n instead of $u^{k+1}_{i,m+n}$, a^k_{m+n}, and v^k_{m+n}, respectively, and writing $\varphi_k(n)$ instead of $\varphi_k(m+n)$, all properties proved above remain valid, and, in addition, the following hold true:

$$|u^{k+1}_{1,n} - u^{k+1}_{2,n}|_p \leq \frac{1}{k+1} \tag{3.17}$$

and

$$|\mathcal{F}(u^{k+1}_{1,n}) - \mathcal{F}(u^{k+1}_{2,n})|_\nu \geq C_2 - \frac{1}{k+1} \tag{3.18}$$

for all $n \in \mathbb{N}$ and

$$\left\| \sum_{j=\ell}^{k} a^j_{\psi^k_j(n)} \star v^j_{\psi^k_j(n)} \right\|^2 \leq 2 \sum_{j=\ell}^{k} \|v^j\|^2, \qquad \text{for all } n \in \mathbb{N} \text{ and } \ell \leq k. \tag{3.19}$$

Now we consider the process of constructing sequences as finished and proceed to prove properties of the whole set. By induction, (3.13) leads to

$$\|u_{1,n}^{k+1}\|^2 = \|u^0_{1,\psi^k_{-1}(n)}\|^2 - \sum_{j=0}^{k} \|v^j\|^2 + o(1), \qquad \text{as } n \to \infty,$$

and hence $\sum_{j=0}^{\infty} \|v^j\|^2 \leq C_1$ by (3.4). In view of (3.9) this yields

$$q_k \to 0, \qquad \text{as } k \to \infty. \tag{3.20}$$

We claim that the diagonal sequence $(u^n_{1,n})$ satisfies

$$b_n \star u^n_{1,n} \rightharpoonup 0, \qquad \text{in } H^1, \text{ as } n \to \infty, \text{ for every sequence } (b_n) \subseteq \mathbb{Z}. \tag{3.21}$$

Note that by construction, for all $\ell \leq k$

$$u^k_{1,n} = u^\ell_{1,\psi^{k-1}_{\ell-1}(n)} - \sum_{j=\ell}^{k-1} a^j_{\psi^{k-1}_j(n)} \star v^j_{\psi^{k-1}_j(n)}.$$

Hence we have the representation

$$u^n_{1,n} = u^k_{1,\psi^{n-1}_{k-1}(n)} - \sum_{j=k}^{n-1} a^j_{\psi^{n-1}_j(n)} \star v^j_{\psi^{n-1}_j(n)}, \qquad \text{if } n \geq k. \tag{3.22}$$

First we show that

$$\operatorname*{w-lim}_{n\to\infty} (-a^k_{\psi^{n-1}_k(n)}) \star u^n_{1,n} = 0, \qquad \text{in } H^1, \text{ for all } k \in \mathbb{N}_0. \tag{3.23}$$

Fix $k \in \mathbb{N}_0$. For every $w \in H^1$ and $\varepsilon > 0$ there is $\ell_0 \geq k+1$ such that

$$\|w\|^2 \sum_{j=\ell_0}^{\infty} \|v^j\|^2 \leq \varepsilon^2/2.$$

Then (3.19), (3.22), and the translation invariance of the norm yield for

$n \geq \ell_0$

$$\begin{aligned}
&\left|\left\langle (-a^k_{\psi_k^{n-1}(n)}) \star u^n_{1,n}, w\right\rangle\right| \\
&\qquad \leq \left|\left\langle (-a^k_{\psi_k^{n-1}(n)}) \star u^{k+1}_{1,\psi_k^{n-1}(n)}, w\right\rangle\right| \\
&\qquad\qquad + \left|\left\langle \sum_{j=k+1}^{\ell_0-1} (a^j_{\psi_j^{n-1}(n)} - a^k_{\psi_k^{n-1}(n)}) \star v^j_{\psi_j^{n-1}(n)}, w\right\rangle\right| \\
&\qquad\qquad + \|w\| \left\| \sum_{j=\ell_0}^{n-1} a^j_{\psi_j^{n-1}(n)} \star v^j_{\psi_j^{n-1}(n)} \right\| \\
&\qquad \leq \left|\left\langle (-a^k_{\psi_k^{n-1}(n)}) \star u^{k+1}_{1,\psi_k^{n-1}(n)}, w\right\rangle\right| \\
&\qquad\qquad + \left|\left\langle \sum_{j=k+1}^{\ell_0-1} (a^j_{\psi_j^{n-1}(n)} - a^k_{\psi_k^{n-1}(n)}) \star v^j_{\psi_j^{n-1}(n)}, w\right\rangle\right| + \varepsilon.
\end{aligned}$$

It is easy to see that the sequence $(\psi_k^{n-1}(n))_n$ is strictly increasing. Hence the first term in the last expression tends to 0 as $n \to \infty$ by (3.7), and the second term tends to 0 by (3.10) and (3.16). Since $\varepsilon > 0$ and $w \in H^1$ were arbitrary, this proves (3.23).

To finish the proof of (3.21), suppose for a contradiction that $\text{w-lim}_{n\to\infty} b_n \star u^n_{1,n} = v \neq 0$ in H^1, for a subsequence. Equation (3.23) implies that

$$\lim_{n\to\infty} \left| b_n + a^k_{\psi_k^{n-1}(n)} \right| = \infty,$$

for every $k \in \mathbb{N}_0$. Pick $k \in \mathbb{N}_0$ such that $q_k < \|v\|$. This is possible by (3.20). Then, for every $w \in H^1$, it follows from (3.19) and (3.22) that

$$\begin{aligned}
&\left|\left\langle b_n \star u^k_{1,\psi_{k-1}^{n-1}(n)} - v, w\right\rangle\right| \\
&\qquad \leq \left|\left\langle b_n \star u^n_{1,n} - v, w\right\rangle\right| + \left|\left\langle \sum_{j=k}^{n-1} (b_n + a^j_{\psi_j^{n-1}(n)}) \star v^j_{\psi_j^{n-1}(n)}, w\right\rangle\right| \to 0
\end{aligned}$$

as $n \to \infty$, similarly as above. Hence

$$\underset{n\to\infty}{\text{w-lim}} \left(b_n \star u^k_{1,\psi_{k-1}^{n-1}(n)} \right) = v$$

with $\|v\| > q_k$. Since $\left(u^k_{1,\psi^{n-1}_{k-1}(n)}\right)_n$ is a subsequence of $(u^k_{1,n})_n$, this contradicts the definition of q_k and proves (3.21).

We are now in the position to finish the proof of uniform continuity of $\mathcal{F}$. Equations (3.17) and (3.18) imply that

$$\lim_{n\to\infty} |u^n_{1,n} - u^n_{2,n}|_p = 0 \tag{3.24}$$

and

$$\liminf_{n\to\infty} |\mathcal{F}(u^n_{1,n}) - \mathcal{F}(u^n_{2,n})|_\nu \geq C_2. \tag{3.25}$$

By Lemma 3.1 and (3.21) $u^n_{1,n} \to 0$ in L^p. Together with (3.24) and (3.25) this contradicts the continuity of $\mathcal{F}$ on L^p and therefore proves the assertion about uniform continuity.

It only remains to prove BL-splitting for $\mathcal{F}$ along weakly convergent sequences in H^1 with respect to their weak limits. Suppose that $u_n \rightharpoonup v$ in H^1. By Theorem 2.1(b) there is a sequence $(v_n) \subseteq H^1$ such that $v_n \to v$ in H^1 and, after passing to a subsequence of (u_n),

$$\mathcal{F}(u_n) - \mathcal{F}(u_n - v_n) \to \mathcal{F}(v), \qquad \text{in } L^\nu \tag{3.26}$$

as $n \to \infty$. Since (u_n) and (v_n) are bounded in H^1, and by the uniform continuity of $\mathcal{F}$ on bounded subsets of H^1 with respect to the L^p-norm (and hence also with respect to the H^1-norm), it follows that we may replace v_n by v in (3.26). Using this, a standard reasoning by contradiction yields the claim. □

4 Construction of Examples

Proof of Example 1.2. We first treat the case $f(t) := \cos(\pi t)|t|^p$. Set $R_n := n^{-p/N}$ and fix a sequence $(x_n) \subseteq \mathbb{R}^N$ such that $|x_n| \to \infty$ and $B_{R_m}(x_m) \cap B_{R_n}(x_n) = \varnothing$ for $m \neq n$. Define real functions u and u_n on $\mathbb{R}^N$ by setting

$$u := \sum_{k=1}^\infty \chi_{B_{R_k}(x_k)}, \qquad w_n := 2n\chi_{B_{R_n}(x_n)}, \qquad \text{and} \qquad u_n := u + w_n,$$

for each $n \in \mathbb{N}$. It is straightforward to show that $u \in L^p$, that (u_n) is a bounded sequence in L^p, and that $u_n \to u$ pointwise and in L^p_{loc}. On the other hand, denoting by ω_N the volume of the unit ball in $\mathbb{R}^N$, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} f(u_n) - f(u_n - u) - f(u)\right| \\
&\qquad = \left|\int_{B_{R_n}(x_n)} f(u + w_n) - f(w_n) - f(u)\right| \\
&\qquad = \left|\int_{B_{R_n}(x_n)} \cos((2n+1)\pi)(2n+1)^p - \cos(2n\pi)(2n)^p - \cos\pi\right| \\
&\qquad = \left|\int_{B_{R_n}(x_n)} (-(2n+1)^p - (2n)^p + 1)\right| \\
&\qquad = \omega_N \left(\left(2 + \frac{1}{n}\right)^p + 2^p - \left(\frac{1}{n}\right)^p\right) \\
&\qquad \to 2^{p+1}\omega_N,
\end{aligned} \tag{4.1}$$

as $n \to \infty$. Since $2^{p+1}\omega_N > 0$, this implies the claim.

For the other example, $f(t) := \cos(\pi/t)|t|^p$, we set $R_n := n^{p/N}$ and fix a sequence $(x_n) \subseteq \mathbb{R}^N$ such that $|x_n|/R_n \to \infty$ and $B_{R_m}(x_m) \cap B_{R_n}(x_n) = \varnothing$ for $m \neq n$. We define

$$u := \sum_{k=1}^{\infty} \frac{1}{2n(2n-1)} \chi_{B_{R_k}(x_k)}, \qquad w_n := \frac{1}{2n} \chi_{B_{R_n}(x_n)}, \qquad \text{and} \qquad u_n := u + w_n$$

for each $n \in \mathbb{N}$. Then again, $u \in L^p$, (u_n) is a bounded sequence in L^p, and $u_n \to u$ pointwise and in L^p_{loc}. For $x \in B_{R_n}(x_n)$ we obtain

$$u(x) + w_n(x) = \frac{1}{2n(2n-1)} + \frac{1}{2n} = \frac{1}{2n-1} \tag{4.2}$$

and hence

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} f(u_n) - f(u_n - u) - f(u)\right| \\
&\quad = \left|\int_{B_{R_n}(x_n)} f(u + w_n) - f(w_n) - f(u)\right| \\
&\quad \geq \left|\int_{B_{R_n}(x_n)} \cos((2n-1)\pi)\left(\frac{1}{2n-1}\right)^p - \cos(2n\pi)\left(\frac{1}{2n}\right)^p\right| \\
&\qquad\qquad\qquad\qquad - \int_{B_{R_n}(x_n)} \left(\frac{1}{2n(2n-1)}\right)^p \\
&\quad = \left|\int_{B_{R_n}(x_n)} -\left(\frac{1}{2n-1}\right)^p - \left(\frac{1}{2n}\right)^p\right| - \int_{B_{R_n}(x_n)} \left(\frac{1}{2n(2n-1)}\right)^p \\
&\quad = \omega_N \left(\left(\frac{1}{2-\frac{1}{n}}\right)^p + \left(\frac{1}{2}\right)^p - \left(\frac{1}{2(2n-1)}\right)^p\right) \\
&\quad \to \frac{\omega_N}{2^{p-1}},
\end{aligned}
\tag{4.3}
$$

as $n \to \infty$. This yields the claim. □

Proof of Remark 1.4. The construction of these counterexamples is closely related to Example 1.2. First consider the function $f(t) := \cos(\pi/t)t^2$. We define the Lipschitz-continuous cut-off function $\eta\colon \mathbb{R} \to \mathbb{R}$ by

$$
\eta(t) := \begin{cases} 1, & t \leq 0, \\ 1 - t, & 0 < t < 1, \\ 0, & 1 \leq t, \end{cases}
$$

introduce $R_n := n^{2/N}$, pick a sequence $(x_n) \subseteq \mathbb{R}^N$ such that $|x_n|/R_n \to \infty$ and $B_{R_m+1}(x_m) \cap B_{R_n+1}(x_n) = \varnothing$ for $m \neq n$, and define

$$
u(x) := \sum_{k=1}^{\infty} \frac{1}{2n(2n-1)}\eta(|x - x_k| - R_k), \qquad w_n(x) := \frac{1}{2n}\eta(|x - x_n| - R_n),
$$

and $u_n := u + w_n$ for each $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. It is straightforward to check that $u, w_n \in H^1$ and that (w_n) is bounded in H^1. Since $w_n \to 0$ a.e.,

$w_n \rightharpoonup 0$ in H^1. Using (4.2) we estimate

$$\begin{aligned}
&\int_{B_{R_n+1}(x_n)\setminus B_{R_n}(x_n)} \Big|f(u+w_n) - f(w_n) - f(u)\Big| \\
&\qquad \le \int_{B_{R_n+1}(x_n)\setminus B_{R_n}(x_n)} \left(\left(\frac{1}{2n-1}\right)^2 + \left(\frac{1}{2n}\right)^2 + \left(\frac{1}{2n(2n-1)}\right)^2\right) \\
&\qquad\qquad \le \frac{3\omega_N}{n^2}((R_n+1)^N - R_n^N) = 3\omega_N((1+n^{-2/N})^N - 1) \to 0
\end{aligned}$$

as $n \to \infty$. Hence by the calculation in (4.3)

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} f(u_n) - f(u_n - u) - f(u)\Big| \\
&\qquad = \Big|\int_{B_{R_n+1}(x_n)} f(u+w_n) - f(w_n) - f(u)\Big| \\
&\qquad \ge \Big|\int_{B_{R_n}(x_n)} f(u+w_n) - f(w_n) - f(u)\Big| \\
&\qquad\qquad\qquad - \int_{B_{R_n+1}(x_n)\setminus B_{R_n}(x_n)} \Big|f(u+w_n) - f(w_n) - f(u)\Big| \\
&\qquad \to \frac{\omega_N}{2^{p-1}}
\end{aligned}$$

and the claim follows. Note that the example above has no simple analogue in the case $f(t) := \cos(\pi/t)|t|^p$ for $p > 2$, using $R_n := n^{p/N}$ as in the proof of the second case of Example 1.2. The reason is that the analogously defined sequence (w_n) is not bounded in L^2 in that case.

Now we treat the function $f(t) := \cos(\pi t)|t|^{2^*}$. To this end put $R_n := n^{-2^*/N}$, fix a sequence $(x_n) \subseteq \mathbb{R}^N$ such that $|x_n| \to \infty$ and $B_{2R_m}(x_m) \cap B_{2R_n}(x_n) = \varnothing$ for $m \neq n$, and choose $\gamma \in (0,1)$ small enough such that

$$\Big(3^{2^*} + 2^{2^*} + 1\Big)\Big((1+\gamma)^N - 1\Big) \le \frac{2^{2^*+1}}{2}. \tag{4.4}$$

Define

$$u(x) := \sum_{k=1}^{\infty} \eta\bigg(\frac{|x-x_k| - R_k}{\gamma R_k}\bigg), \qquad w_n(x) := 2n\eta\bigg(\frac{|x-x_k| - R_n}{\gamma R_n}\bigg),$$

and $u_n := u + w_n$ for each $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. It follows that $\operatorname{supp}(w_n) = \overline{B}_{(1+\gamma)R_n}(x_n)$ for each n, where $\overline{B}_r(z)$ denotes the closed ball in $\mathbb{R}^N$ with radius r and center z. Again, it is straightforward to check that $u, w_n \in H^1$, that (w_n) is bounded in H^1, and that $w_n \rightharpoonup 0$ in H^1. Using (4.4) we estimate

$$\begin{aligned}
&\int_{B_{(1+\gamma)R_n}(x_n)\setminus B_{R_n}(x_n)} \Big| f(u+w_n) - f(w_n) - f(u) \Big| \\
&\qquad \le \int_{B_{(1+\gamma)R_n}(x_n)\setminus B_{R_n}(x_n)} \Big((1+2n)^{2^*} + (2n)^{2^*} + 1 \Big) \\
&\qquad \le \omega_N \Big((3n)^{2^*} + (2n)^{2^*} + n^{2^*} \Big) \Big(((1+\gamma)R_n)^N - R_n^N \Big) \\
&\qquad = \omega_N \Big(3^{2^*} + 2^{2^*} + 1 \Big) \Big((1+\gamma)^N - 1 \Big) \\
&\qquad \le \frac{2^{2^*+1}\omega_N}{2}
\end{aligned}$$

for all n. Hence by the calculation in (4.1)

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} f(u_n) - f(u_n - u) - f(u) \Big| \\
&\quad = \Big| \int_{B_{(1+\gamma)R_n}(x_n)} f(u+w_n) - f(w_n) - f(u) \Big| \\
&\quad \ge \Big| \int_{B_{R_n}(x_n)} f(u+w_n) - f(w_n) - f(u) \Big| \\
&\qquad\qquad - \int_{B_{(1+\gamma)R_n}(x_n)\setminus B_{R_n}(x_n)} \Big| f(u+w_n) - f(w_n) - f(u) \Big| \\
&\quad \ge \Big| \int_{B_{R_n}(x_n)} f(u+w_n) - f(w_n) - f(u) \Big| - \frac{2^{2^*+1}\omega_N}{2} \\
&\quad \to \frac{2^{2^*+1}\omega_N}{2}
\end{aligned}$$

and the claim follows. Note that this example has no simple analogue in the case $f(t) := \cos(\pi t)|t|^p$ for $p < 2^*$, using $R_n := n^{-p/N}$ as in the proof of the first case of Example 1.2. Here the reason is that for the analogously defined sequence (w_n), (∇w_n) is not bounded in L^2. □

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References

[1] N. Ackermann, *On a periodic Schrödinger equation with nonlocal superlinear part*, Math. Z. **248** (2004), no. 2, 423–443. MR MR2088936 (2005i:35075)

[2] ______, *A nonlinear superposition principle and multibump solutions of periodic Schrödinger equations*, J. Funct. Anal. **234** (2006), no. 2, 277–320. MR MR2216902

[3] N. Ackermann and T. Weth, *Multibump solutions of nonlinear periodic Schrödinger equations in a degenerate setting*, Commun. Contemp. Math. **7** (2005), no. 3, 269–298. MR MR2151860

[4] J. Appell and P.P. Zabrejko, *Nonlinear superposition operators*, Cambridge Tracts in Mathematics, vol. 95, Cambridge University Press, Cambridge, 1990. MR MR1066204 (91k:47168)

[5] Y.V. Bogdanskii, *Laplacian with respect to a measure on a Hilbert space and an L_2-version of the Dirichlet problem for the Poisson equation*, Ukrainian Math. J. **63** (2012), no. 9, 1336–1348. MR 3109657

[6] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490. MR 699419 (84e:28003)

[7] J. Chabrowski, *Weak convergence methods for semilinear elliptic equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1999. MR 1885519 (2003a:35056)

[8] G. Da Prato and J. Zabczyk, *Second order partial differential equations in Hilbert spaces*, London Mathematical Society Lecture Note Series, vol. 293, Cambridge University Press, Cambridge, 2002. MR 1985790

[9] ———, *Stochastic equations in infinite dimensions*, second ed., Encyclopedia of Mathematics and its Applications, vol. 152, Cambridge University Press, Cambridge, 2014. MR 3236753

[10] Y. Ding and A. Szulkin, *Bound states for semilinear Schrödinger equations with sign-changing potential*, Calc. Var. Partial Differential Equations **29** (2007), no. 3, 397–419. MR MR2321894 (2008c:35067)

[11] Y. Ding and J. Wei, *Semiclassical states for nonlinear Schrödinger equations with sign-changing potentials*, J. Funct. Anal. **251** (2007), no. 2, 546–572. MR 2356423

[12] L. Gross, *Potential theory on Hilbert space*, J. Functional Analysis **1** (1967), 123–181. MR 0227747

[13] W. Kryszewski and A. Szulkin, *Infinite-dimensional homology and multibump solutions*, J. Fixed Point Theory Appl. **5** (2009), no. 1, 1–35. MR MR2491945

[14] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 2, 109–145. MR 778970 (87e:49035a)

[15] ———, *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), no. 4, 223–283. MR 778974 (87e:49035b)

[16] ———, *Solutions of Hartree-Fock equations for Coulomb systems*, Comm. Math. Phys. **109** (1987), no. 1, 33–97. MR 88e:35170

[17] E. Priola, *On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions*, Studia Math. **136** (1999), no. 3, 271–295. MR 1724248

[18] K. Tintarev and K.H. Fieseler, *Concentration compactness*, Imperial College Press, London, 2007, Functional-analytic grounds and applications. MR 2294665 (2009b:46074)

[19] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1400007 (97h:58037)