A Nonlinear Superposition Principle and Multibump Solutions of Periodic Schrödinger Equations

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Abstract

In an abstract setting we prove a nonlinear superposition principle for zeros of equivariant vector fields that are asymptotically additive in a well-defined sense. This result is used to obtain multibump solutions for two basic types of periodic stationary Schrödinger equations with superlinear nonlinearity. The nonlinear term may be of convolution type. If the superquadratic term in the energy functional is convex, our results also apply in certain cases if 0 is in a gap of the spectrum of the Schrödinger operator.

1. Introduction and Results

For N in $\mathbb{N}$ and a Caratheodory function $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ consider

$$-\Delta u + V(x)u = f(x,u) \qquad u \in H^1(\mathbb{R}^N)\ . \tag{L}$$

We assume that $f(x,u)/u \to \infty$ as $|u| \to \infty$, that $f(x,0) = 0$, and that V and f are periodic with respect to x. Let T denote the unique self adjoint operator induced on $L^2(\mathbb{R}^N)$ by $-\Delta + V$. Denote by $\sigma(T)$ the $L^2(\mathbb{R}^N)$-spectrum of T. We assume that $0 \notin \sigma(T)$ and let the *positive case* (or the case of *mountain pass geometry*) refer to $\sigma(T) \subseteq (0,\infty)$, and the *strongly indefinite case* (or the case of *strongly indefinite geometry*) to $\sigma(T) \cap (-\infty,0) \neq \varnothing$.

Let us first recall some known results. Existence of a nontrivial solution of (L) is shown for the positive case in [38]. In the strongly indefinite case existence results are given in [3, 25] under the assumption that f increases strictly in u. Without this extra assumption existence results for the strongly indefinite case can be found in [27, 36, 44]. The two papers [9, 46] treat the case where 0 is the left endpoint of a gap of the spectrum of T (see also [10]). Related results are also contained in [17–22, 28, 43], where equations are treated that are not fully periodic in x.

For certain periodically forced Hamiltonian systems infinitely many homoclinic orbits are constructed as multibump solutions in [40] and [15]. Using some ideas from these papers, in [4, 16, 41] it is shown for the positive case that (L) possesses infinitely many geometrically distinct solutions. The article [4] also covers the strongly indefinite case (for a very restricted class of nonlinearities f) and asymptotically periodic equations. Our result in [2] gives a multiplicity result under very weak differentiability hypotheses on f. All of these multiplicity results are proved by constructing multibump solutions. In [2,16,32,33] also nodal properties of multibump solutions are considered. The earliest reference we are aware of where multibump solutions are constructed for an elliptic PDE is [8] (here it was done under different assumptions). More references on multibump solutions can be found in the survey article [39] with focus on homoclinic orbits of Hamiltonian systems. For f odd in u multiplicity of solutions for (L) in the positive and the strongly indefinite case is shown in [9,27] (see also [10]). In contrast to the multibump results mentioned above, in the latter references the authors develop a global variational approach, applying a suitable index theory.

Motivated by the difficulty to adapt the methods used in [4, 16] to equations with nonlocal terms, our goal in the present paper is to provide an abstract framework in which multibump solutions can be obtained in many situations. The main result here is Theorem 3.4. It reduces the problem of constructing multibump solutions to the problem of finding an isolated solution with nontrivial topology in a specific sense.

We now describe our new results with respect to applications. Consider the following class of nonlocal equations in $\mathbb{R}^3$:

$$\text{(NL)} \qquad -\Delta u + V(x)u = (W * u^2)u \qquad u \in H^1(\mathbb{R}^3)\ .$$

The function $W\colon \mathbb{R}^3 \to [0,\infty)$ is assumed to be measurable and to lie in some suitable function space (see Section 1.2) such that $W * u^2$, the convolution of the functions W and u^2, is well defined for $u \in H^1(\mathbb{R}^3)$.

This equation is treated for $W(x) = 1/|x|$ in [12], where it is shown that (NL) admits a nontrivial solution. Note that the proof extends to the case that W is a positive definite function with suitable growth restrictions. Roughly, a positive definite function is a function with nonnegative Fourier transform (in the sense of distributions). For a survey on the notion of positive definite functions c.f. [42]. In [1] the existence result is derived without the assumption of positive definiteness of W, for a more general class of equations. In the latter paper, using results from [9, 10], also multiplicity of solutions for (NL) is proved.

Our first result concerning applications is that (NL) admits multibump solutions. It is contained in Theorem 1.2 below. To the best of our knowledge existence of multibump solutions has not been shown before for (NL).

Even though we were initially interested in nonlocal problems we also obtain a new result for the local problem: From Theorem 1.2 it follows that multibump solutions for (L) exist in the strongly indefinite case if f is strictly increasing in u. We show this for a much broader class of functions f than considered in [4], see assumption (A1.4) below.

1.1. Assumptions on the Local Problem (L)

Denote by $2^* := \infty$ for $N = 1, 2$ and $2^* := 2N/(N-2)$ for $N \geq 3$ the critical Sobolev exponent. Recall that we have set $T = -\Delta + V$. Using the notation $F(x,u) := \int_0^u f(x,s)\,\mathrm{d}s$ we make the following assumptions:

(A1.1) $V \in L^\infty(\mathbb{R}^N)$, V is periodic, separately in each coordinate direction with minimal period 1, and $0 \notin \sigma(T)$.

(A1.2) $f_u(x,u)$ exists everywhere, and $f_{uu}(x,u)$ exists for $u \neq 0$. f_u is a Caratheodory function. $f(x,0) = f_u(x,0) = 0$ for all x. There are $C \geq 0$ and $p_1, p_2 \in (2, 2^*)$ with $p_1 \leq p_2$ such that

$$|f_{uu}(x,u)| \leq C(|u|^{p_1-3} + |u|^{p_2-3}) \tag{1.1}$$

holds for every $u \neq 0$ and all x. f is periodic in the first argument, separately in each coordinate direction with minimal period 1.

(A1.3) There is $\theta > 2$ such that

$$f(x,u)u \geq \theta F(x,u) > 0$$

holds for every $u \neq 0$ and for all x.

(A1.4) For every $u \neq 0$ and for all x it holds that

$$f_u(x,u)u^2 > f(x,u)u\ .$$

Set $E := H^1(\mathbb{R}^N)$ and define $\Phi\colon E \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x\ .$$

By our assumptions $\Phi \in C^2$, and critical points of Φ are in one to one correspondence with weak solutions of (L). Note that by (1.1) the second differential of Φ is also Hölder continuous, at the same time allowing for different superlinear growth of f in u at zero and at infinity.

1.2. Assumptions on the Nonlocal Problem (NL)

We assume (A1.1) and the following hypotheses:

(A1.5) There is $r \in [1, \infty)$ such that $W \in L^1(\mathbb{R}^3) + L^r(\mathbb{R}^3)$, and W is even.

(A1.6) $W \geq 0$, and $W > 0$ on a neighborhood of 0.

(A1.7) $\sigma(T) \subseteq (0, \infty)$ or W is positive definite.

Set $E := H^1(\mathbb{R}^3)$ and define $\Phi\colon E \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} (W * u^2)u^2\,\mathrm{d}x\ .$$

By our assumptions $\Phi \in C^2$ and critical points of Φ are in one to one correspondence with weak solutions of (NL).

1.3. Results for the Applications

We treat (L) and (NL) together in the unified setting we have established so far. Note that $\Phi(0) = 0$ and $\Phi'(0) = 0$. Define

$$\mathcal{K} := \{\, u \in E \setminus \{0\} \mid \Phi'(u) = 0 \,\} \ .$$

Observe that for convenience we have excluded the critical point 0 from $\mathcal{K}$. It is known that, under the above assumptions, 0 is isolated in the set of critical points of Φ, so $\mathcal{K}$ is closed. For all c, d in $\mathbb{R}$ denote

$$\begin{aligned} \mathcal{K}^c &:= \{\, u \in \mathcal{K} \mid \Phi(u) \le c \,\} \\ \mathcal{K}_c^d &:= \{\, u \in \mathcal{K} \mid c \le \Phi(u) \le d \,\} \\ \mathcal{K}(c) &:= \mathcal{K}_c^c \ . \end{aligned}$$

A sequence (u_n) in E with $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ is called a Palais-Smale sequence at the level c, or $(\mathrm{PS})_c$-sequence in short.

Denote by $\star$ the action of $\mathbb{Z}^N$ on E that arises from translation: For $u \in E$ and $a \in \mathbb{Z}^N$ define $(a \star u)(x) := u(x - a)$. From the periodicity assumptions on V (and f in (L)) it follows that Φ is invariant under the action of $\mathbb{Z}^N$, so $\mathbb{Z}^N$ also acts on $\mathcal{K}$.

1.1 Definition. Two elements u, v of E will be called *geometrically distinct* if u and v do not belong to the same class of $E/\mathbb{Z}^N$.

We say that a finite subset A of $\mathcal{K}$ *generates multibump critical points of* Φ if for every $k \in \mathbb{N}$, $\varepsilon > 0$ there is $M \ge 0$ such that for all $a_i \in \mathbb{Z}^N$ and $u_i \in A$ $(i = 1, 2, \ldots, k)$ with

$$|a_i - a_j| \ge M \qquad \text{if } i \ne j$$

there is $u \in \mathcal{K}$ such that

$$\begin{aligned} \left\| u - \sum_{i=1}^{k} a_i \star u_i \right\| &\le \varepsilon \\ \left| \Phi(u) - \sum_{i=1}^{k} \Phi(u_i) \right| &\le \varepsilon \ . \end{aligned}$$

We also call such a critical point u a (weak) *multibump solution of* (L) *respectively* (NL).

It is well known by now that under our assumptions $\mathcal{K} \ne \varnothing$ and that Φ attains a positive minimum on $\mathcal{K}$. Therefore we set

$$c_{\min} := \min \Phi(\mathcal{K}) > 0 \ .$$

We call a critical point of Φ *isolated* if it is isolated in the set $\mathcal{K}$. For the applications, our main result then reads:

1.2 Theorem. a) *Every finite set of isolated critical points in $\mathcal{K}(c_{\min})$ generates multibump critical points of Φ.*

b) *If there is an isolated critical point in* $\mathcal{K}(c_{\min})$ *then for every* $k \in \mathbb{N} \setminus \{1\}$ *and* $\varepsilon > 0$ *the set* $\mathcal{K}_{kc_{\min}-\varepsilon}^{kc_{\min}+\varepsilon}/\mathbb{Z}^N$ *is infinite.*

If $\mathcal{K}^{c_{\min}+\varepsilon}/\mathbb{Z}^N$ is finite for some $\varepsilon > 0$ then all elements in $\mathcal{K}(c_{\min})$ are isolated critical points of Φ. Hence the theorem states that under our assumptions there are always infinitely many geometrically distinct weak solutions for both of (L) and (NL).

1.4. Discussion

Our main abstract result, Theorem 3.4, can be viewed as a nonlinear superposition principle for zeros of equivariant vector fields that are asymptotically additive, in a sense made precise via the notion of a *BL-splitting* map (cf. Definition 3.1 and condition (F3.2)). Starting with certain isolated zeros as building blocks, one obtains new zeros near the sum of their translates if the translates are sufficiently far apart from each other. For this principle to be applicable, some nontrivial topological information on the building blocks is needed, namely nonvanishing of the local degree of the vector field (after a finite-dimensional local reduction). Theorem 3.4 is essentially independent of any variational structure. Nevertheless, for simplicity we assume that the vector field is the gradient of some functional. Symmetry of the derivative facilitates various constructions and estimates.

For the application to the variational setting introduced above we consider the equivariant gradient vector field of Φ. We obtain the nontriviality of the reduced local degree from the local linking structure of critical points. In the positive case these are points of mountain pass type. The strongly indefinite case poses a harder problem. We are only able to treat it here assuming convexity of the superquadratic part of Φ. This enables us to reduce the problem to one with mountain pass geometry, and then to proceed as in the positive case.

It is a challenge to prove a similar superposition principle from weaker topological information on isolated critical points, for example from the existence of a nontrivial critical group. Another interesting open problem is to remove the convexity assumption, namely that f is increasing in u with respect to problem (L), in the strongly indefinite case.

Let us comment on the hypotheses described in Sections 1.1 and 1.2. Concerning assumption (A1.2) we remark that in our proof it is only needed that the superposition operator induced by f_u is uniformly Hölder continuous on bounded subsets of E.

In the positive case from [16] it is known that condition (A1.4) is not needed to construct multibump solutions for (L). However, as mentioned above, our proof relies on the fact that the local degrees of certain isolated critical points of mountain pass type are not zero (after a finite-dimensional reduction). To show this requires that the kernel of Φ'' at such points is 1-dimensional if the Morse index vanishes. Assumption (A1.4) implies that every critical point at the level $c_{\min}$ is of mountain pass type with Morse index 1, hence satisfying the above requirement if it is isolated.

In a forthcoming paper we hope to weaken (A1.2) to the case that f is only once continuously differentiable in u (with appropriate bounds on f_u). Moreover we plan to handle the positive case without assuming (A1.4).

We have restricted our attention here to a very specific nonlocal equation. Using the results from [1] it is easy to apply Theorem 1.2 to a larger class of nonlocal equations with similar structure as (NL).

The organization of the paper is as follows: In Section 2 we analyze the reduction of a vector field at a zero to the kernel of the differential and introduce the notion of reduced local degree. Section 3 contains the statement and proof of the nonlinear superposition principle. In Section 4 we show that the reduced degree of isolated critical points with minimal positive energy is nonzero in the case of mountain pass geometry. Reducing it to mountain pass geometry, we deal with the strongly indefinite case in Section 5. Finally, in Section 6 we show that assumptions (A1.1)–(A1.7) are sufficient for the application of the abstract results to (L) and (NL).

For the convenience of the reader there is a list of extra notation (used in Sections 2–5) included in Table 1.

1.5. General Notation

We set $\mathbb{R}^+ := (0,\infty)$, $\mathbb{R}_0^+ := [0,\infty)$, $\mathbb{R}^- := (\infty,0)$, and $\mathbb{R}_0^- := (\infty,0]$. For $q \in [1,\infty]$ we denote the norm in $L^q(\mathbb{R}^N)$ by $|\cdot|_q$. The scalar product in $L^2(\mathbb{R}^N)$ is written as $(\cdot\,,\cdot)$. Initially we endow $H^1(\mathbb{R}^N)$ with the scalar product

$$\langle u,v\rangle_{H^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (\nabla u\cdot\nabla v + uv)\,\mathrm{d}x$$

and the associated norm $\|u\|_{H^1(\mathbb{R}^N)} := \sqrt{|\nabla u|_2^2 + |u|_2^2}$.

If X is a metric space, A is a point or a subset of X, and $\rho \geq 0$, then we set

$$\begin{aligned}
U(\rho,A;X) &:= U_\rho(A;X) := \{\, x\in X \mid \operatorname{dist}_X(x,A) < \rho \,\}\\
B(\rho,A;X) &:= B_\rho(A;X) := \{\, x\in X \mid \operatorname{dist}_X(x,A) \leq \rho \,\}\\
S(\rho,A;X) &:= S_\rho(A;X) := \{\, x\in X \mid \operatorname{dist}_X(x,A) = \rho \,\}\ .
\end{aligned}$$

When there is no confusion possible we usually omit the X-dependency. If $(X,\|\cdot\|)$ is a normed vector space and $A=0$, we often write $U_\rho X$ instead of $U_\rho(0;X)$, and so forth. Also in this case we may omit the X-dependency.

For normed vector spaces X,Y we denote by $\mathcal{L}(X,Y)$ the space of bounded linear maps from X to Y, endowed with the uniform operator norm, and by $\mathcal{L}_s(X,Y)$ the same space endowed with the strong operator topology. As usual, if $X=Y$ we write $\mathcal{L}(X) := \mathcal{L}(X,X)$. The dual of X is denoted by X^*, and the adjoint of A in $\mathcal{L}(X,Y)$ is denoted by A^*. The space X_w is the space X endowed with its weak topology. We denote weak convergence of a sequence in X with the symbol $\rightharpoonup$. If X,Y are normed spaces and $f\colon X\to Y$ is a map, we say that f is weakly sequentially continuous if $f\colon X_w \to Y_w$ is sequentially continuous.

The kernel of a linear operator A will be denoted by $\mathcal{N}(A)$, its range by $\mathcal{R}(A)$. In a Hilbert space setting the symbol P will be used exclusively for orthogonal projections. Bounded projections that are not orthogonal will be denoted with symbols different from P. Usually the range of a projection is given in the subscript.

If $U \subseteq X$ is open, $n \in \mathbb{N}_0$ and $\alpha \in (0,1)$, we write $C^n(U,Y)$ for the space of functions that have continuous derivatives up to order n, and by $C^{n+\alpha}(U,Y)$ the subspace of functions in $C^n(U,Y)$ where the n-th derivative is locally Hölder continuous with exponent α. By $C^{n-}(U,Y)$ for $n \geq 1$ we denote the subspace of functions in $C^{n-1}(U,Y)$ where the derivative of order $(n-1)$ is locally Lipschitz. We call a map from U into Y *bounded* if it maps bounded subsets of U into bounded subsets of Y. We say that $u \in C^n(U,Y)$ *uniformly on bounded subsets* if all derivatives up to order n are bounded in this sense. For $\alpha \in (0,1)$ we say that $u \in C^{n+\alpha}(U,Y)$ *uniformly on bounded subsets* if $u \in C^n(U,Y)$ uniformly on bounded subsets and if the n-th derivative of u is uniformly Hölder continuous with exponent α on bounded subsets of U. A similar convention applies to spaces of Lipschitz continuous functions.

For a finite-dimensional Banach space X, an open bounded subset U of X, a continuous map $f\colon \overline{U} \to X$, and $y \in X \setminus f(\partial U)$ the mapping degree of f with respect to y is denoted as usual by $\deg(f,U,y)$. If x is an isolated zero of f, the local degree of f at x (index of the zero x) will be denoted by $\deg_{\mathrm{loc}}(f,x)$.

2. Reductions and the Reduced Local Degree

Here we introduce and analyze the notion of a local degree at a zero of a vector field after a suitable finite-dimensional reduction. We do not intend to develop a degree theory. Only some facts needed for the proof and application of the nonlinear superposition principle will be presented.

2.1 Definition. Suppose that Z is a Banach space, and that X, Y are closed subspaces of Z such that $Z = X \oplus Y$. Endow X and Y with the norms induced by the norm of Z. Denote by P_Y the projection in Z onto Y along X, and set $P_X := I - P_Y$. Suppose that for some $z_0 \in Z$ and an open neighborhood U of z_0 in Z we are given $f \in C^1(U,Z)$ such that $P_Y f(z_0) = 0$, and such that $P_Y f'(z_0)|_Y \in \mathcal{L}(Y)$ is an isomorphism. Consider the set

$$W := \{\, z \in U \mid P_Y f(z) = 0 \,\} .$$

Then by the implicit function theorem W can be described near z_0 as the graph of a C^1-map $h\colon V_X \to Y$, where V_X is an open neighborhood of 0 in X, as follows. There is an open neighborhood V_Y of 0 in Y such that

$$W \cap (z_0 + V_X + V_Y) = \{\, z_0 + x + h(x) \mid x \in V_X \,\} .$$

We call the map $g\colon V_X \to X$, given by $g(x) := f(z_0 + x + h(x))$, a *reduction of f at z_0 to X along Y*. Note that if $X = \{0\}$ then g is just the trivial map on X.

If Z is a Hilbert space and $X \perp Y$, then we will usually omit the "along" part and say that g is a *reduction of f at z_0 to X*. If f is the gradient of some C^2-functional $\Phi\colon U \to \mathbb{R}$, define $\Psi(x) := \Phi(z_0 + x + h(x))$. We say that Ψ is a *reduction of Φ at z_0 to X*. In this case g is the gradient of the C^2-functional Ψ.

2.2 Remark. It is clear that, in the setting above, the zeros of f in $z_0 + V_X + V_Y$ are in one-to-one correspondence with the zeros of g in V_X.

2.3 Definition. Suppose that Z is a Banach space and that for some $z_0 \in Z$ and an open neighborhood U of z_0 in Z we are given $f \in C^1(U, Z)$. Suppose moreover that z_0 is an isolated zero of f, that $\sigma(f'(z_0)) \setminus \{0\}$ is closed, and that $X := \mathcal{N}(f'(z_0))$ is finite-dimensional. Note that then $f'(z_0)$ is a Fredholm operator of index 0. Let Y be the closed invariant subspace of Z corresponding to $\sigma(f'(z_0)) \setminus \{0\}$. Then $f'(z_0)|_Y$ is an isomorphism and a reduction g of f at z_0 to X along Y is defined on some neighborhood of 0 in X. Moreover, 0 is an isolated zero of g.

We define the *reduced local degree* $\mathrm{rdeg}_{\mathrm{loc}}(f, z_0)$ *of* f *at* z_0 by $\mathrm{rdeg}_{\mathrm{loc}}(f, z_0) := \deg_{\mathrm{loc}}(g, 0)$. Here we set $\deg_{\mathrm{loc}}(g, 0) := 1$ if $f'(z_0)$ is an isomorphism and hence g is trivial.

If Z is a Hilbert space and f the gradient of some C^2-functional $\Phi\colon U \to \mathbb{R}$ we define the *reduced local degree* $\mathrm{rdeg}_{\mathrm{loc}}(\Phi, z_0)$ of Φ at z_0 by $\mathrm{rdeg}_{\mathrm{loc}}(\Phi, z_0) := \mathrm{rdeg}_{\mathrm{loc}}(f, z_0)$. Note that in this situation for the spectral condition above to hold it suffices to assume that $f'(z_0)$ is Fredholm of index 0 since $f'(z_0)$ is selfadjoint.

We need to have available a quantitative version of the reduction described in Definition 2.1. Moreover, we want to extend Definition 2.1 to the case that z_0 is only an approximate zero of $P_Y f$.

2.4 Lemma. *Let X, Y, Z, P_X and P_Y be given as in Definition* 2.1. *For some constants $\alpha \in (0, 1]$ and $r, M > 0$ suppose that $\|P_Y\| \le M$. Also suppose that we are given $z_0 \in Z$ and $f \in C^{1+\alpha}(B_r(z_0; Z), Z)$ such that*

$$\|f\|_{C^{1+\alpha}(B_r(z_0;Z),Z)} \le M \;,$$

and that $P_Y f'(z_0)|_Y \in \mathcal{L}(Y)$ is an isomorphism with $\|(P_Y f'(z_0)|_Y)^{-1}\| \le M$. Then there are positive constants $r_1 \le r_2 \le r$, C_1 and C_2, only depending on α, r and M, with the following properties: If $\|P_Y f(z_0)\| \le C_1$ and if we set

$$W := \{\, z \in z_0 + B_{r_1} X + B_{r_2} Y \mid P_Y f(z) = 0 \,\}$$

then there is h in $C^{1+\alpha}(B_{r_1} X, Y)$ such that

(i) $W = \{\, z_0 + x + h(x) \mid x \in B_{r_1} X \,\}$

(ii) $\|h\|_{C^{1+\alpha}(B_{r_1} X, Y)} \le C_2$

(iii) $\|h(x)\| \le 2M(M^2\|x\| + \|P_Y f(z_0)\|)$ *for every x in $B_{r_1} X$.*

If we define $g\colon B_{r_1} X \to X$ by

$$g(x) := f(z_0 + x + h(x)) = P_X f(z_0 + x + h(x))$$

then $g \in C^{1+\alpha}(B_{r_1} X, X)$ and we have

(iv) $\|g\|_{C^{1+\alpha}(B_{r_1} X, X)} \le C_2$

(v) $\|g(0) - P_X f(z_0)\| \le C_2 \|P_Y f(z_0)\|$

(vi) $\|g'(0) - P_X f'(z_0)|_X\| \le C_2\Big(\|P_Y f(z_0)\|^\alpha + \|P_X f'(z_0)|_Y\| \cdot \|P_Y f'(z_0)|_X\|\Big)$

In the case that $P_Y f(z_0) = 0$*, we have*

(vii) $h(0) = 0$

(viii) $h'(0) = -(P_Y f'(z_0)|_Y)^{-1} P_Y f'(z_0)|_X$

(ix) $g(0) = f(z_0)$

(x) $g'(0) = P_X f'(z_0)(I_X + h'(0))$

2.5 Remark. In the setting of Lemma 2.4, if $\|P_Y f(z_0)\| \le C_1$ we will say that g is a reduction of f at z_0 to X along Y, therefore widening the scope of Definition 2.1.

Lemma 2.4 can be viewed as a shadowing lemma where a point z_0 that is a "non-degenerate approximate Y-zero" of the vector field f is shadowed by a manifold W of "Y-zeros" transverse to Y. Let us specialize Lemma 2.4 to the case that Y is the whole space Z.

2.6 Corollary. *Suppose that Z is a Banach space. For some constants $\alpha \in (0,1]$ and $r, M > 0$ suppose that we are given $z_0 \in Z$ and $f \in C^{1+\alpha}(B_r(z_0), Z)$ such that*

$$\|f\|_{C^{1+\alpha}(B_r(z_0),Z)} \le M ,$$

and such that $f'(z_0)$ is an isomorphism with $\|(f'(z_0))^{-1}\| \le M$. Then there are positive constants $r_2 \le r$ and C_1, only depending on α, r and M, with the following properties: If $\|f(z_0)\| \le C_1$ then f has exactly one zero z_1 in $B_{r_2}(z_0)$. In addition, $\|z_1 - z_0\| \le 2M\|f(z_0)\|$.

Proof of Lemma 2.4*.* The proof is a simple application of the contraction mapping principle. For convenience of the reader we provide a few details.

Suppose that X, Y, Z and f are given with the properties listed in the statement of the lemma. After translation we may assume that $z_0 = 0$. Set $A := P_X f'(0)|_X \in \mathcal{L}(X)$ and $B := P_Y f'(0)|_Y \in \mathcal{L}(Y)$. We choose $r_2 \le r/2$ independently of X, Y, Z and f such that

$$\|f'(z) - f'(0)\| \le \frac{1}{2M^2} \tag{2.1}$$

for all $z \in B_{2r_2} Z$. Moreover we set $C_1 := r_2/(4M)$ and choose $r_1 \le r_2$ independently of X, Y and f such that

$$\|f(z) - f(0)\| \le \frac{C_1}{M}$$

for all $z \in B_{r_1} Z$. Then if $\|P_Y f(0)\| \le C_1$ it follows that

$$\|P_Y f(z)\| \le 2C_1 = \frac{r_2}{2M} \tag{2.2}$$

for all $z \in B_{r_1}Z$.

Now suppose that $\|P_Y f(0)\| \leq C_1$. Define $\varphi\colon B_{r_1}X \times B_{r_2}Y \to Y$ by

$$\varphi(x,y) := y - B^{-1}P_Y f(x+y) = B^{-1}P_Y(f'(0)y - f(x+y)) \,.$$

It follows that $\varphi(x,y) = y$ if and only if $P_Y f(x+y) = 0$. Moreover

$$\begin{aligned} \|\varphi(x,y)\| &\leq M\|P_Y(f(x+y) - f'(0)y)\| \\ &= M\left\|P_Y f(x) + \int_0^1 P_Y(f'(x+sy) - f'(0))y \,\mathrm{d}s\right\| \\ &\leq M\left(\|P_Y f(x)\| + \frac{1}{2M}\|y\|\right) \\ &\leq r_2 \end{aligned} \tag{2.3}$$

by (2.1) and (2.2), and since $\|x+y\| \leq 2r_2$ and $\|y\| \leq r_2$. So actually φ maps into $B_{r_2}Y$. Now

$$\|D_Y\varphi(x,y)\| \leq M^2\|f'(0) - f'(x+y)\| \leq \frac{1}{2}$$

by (2.1) again. This shows that $\varphi(x,\cdot)$ is a contraction on $B_{r_2}Y$, uniformly in $x \in B_{r_1}X$. From Banach's contraction mapping theorem we obtain a map $h\colon B_{r_1}X \to B_{r_2}Y$ such that $\varphi(x,y) = y$ if and only if $y = h(x)$, for all $(x,y) \in B_{r_1}X \times B_{r_2}Y$. Therefore h satisfies (i). From (2.3) it also follows that

$$\|h(x)\| \leq 2M\|P_Y f(x)\| \leq 2M(M^2\|x\| + \|P_Y f(0)\|)$$

and hence (iii).

Standard arguments [23, 1.2.6] show that h is continuously differentiable. The remaining estimates follow in a straightforward way. □

In the next lemma we relate the local degrees of finite-dimensional reductions of a vector field to different subspaces, one included in the other. Again we define the local degree of the trivial map $\{0\} \to \{0\}$ to be 1.

2.7 Lemma. *Suppose that Z is a Banach space, U is an open neighborhood of z_0 in Z and $f\colon U \to Z$ is C^1. Suppose also that z_0 is an isolated zero of f. Set $A := f'(z_0)$. For $i = 0,1$ let X_i be finite-dimensional and Y_i closed subspaces of Z, such that $Z = X_0 \oplus Y_0 = X_1 \oplus Y_1$, $X_0 \subseteq X_1$, and $Y_1 \subseteq Y_0$. Suppose moreover that X_i, Y_i are invariant with respect to A and that $P_{Y_i}A|_{Y_i}$ are isomorphisms. Denote by g_i a reduction of f at z_0 to X_i along Y_i, so 0 is an isolated zero of g_i. Then*

$$|\deg_{\mathrm{loc}}(g_0,0)| = |\deg_{\mathrm{loc}}(g_1,0)| \,.$$

Proof. We may assume that $z_0 = 0$. Set $Y_2 := Y_0 \cap X_1$ and let g_2 denote a reduction of g_1 at 0 to X_0 along Y_2. We will prove that

$$\deg_{\mathrm{loc}}(g_0,0) = \deg_{\mathrm{loc}}(g_2,0) \tag{2.4}$$

and

$$|\deg_{\mathrm{loc}}(g_1,0)| = |\deg_{\mathrm{loc}}(g_2,0)| \;, \tag{2.5}$$

which proves the claim.

Denote by P_{X_i} the projection onto X_i with kernel Y_i, for $i = 0, 1$ and set $P_{Y_i} := I - P_{X_i}$. Note that $P_{X_0}|_{X_1}$ is the projection in X_1 with range X_0 and kernel Y_2, since $X_1 = X_0 \oplus Y_2$. Denote by

$$\begin{aligned} h_0 &\colon X_0 \to Y_0 \\ h_1 &\colon X_1 \to Y_1 \\ h_2 &\colon X_0 \to Y_2 \end{aligned}$$

the maps defined near 0 that arise from the construction of the respective reductions g_i. It follows that for $x \in X_0$ near 0

$$f(x + h_2(x) + h_1(x + h_2(x))) = g_1(x + h_2(x)) = g_2(x) \in X_0 \;.$$

Therefore by uniqueness $h_2(x) + h_1(x + h_2(x)) = h_0(x)$ and g_0 and g_2 coincide near 0. From this (2.4) follows.

To show (2.5) consider the maps $G \colon X_1 \to X_1$ and $\widetilde{G} \colon X_1 \to \mathcal{L}(X_1)$ defined locally at 0 by

$$G(z) := g_1(x + h_2(x)) + g_1'(0)[y - h_2(x)]$$

and

$$\widetilde{G}(z) := \int_0^1 \Big(g_1'(x + sy + (1 - s)h_2(x)) - g_1'(0) \Big) \,\mathrm{d}s \;.$$

Here and in the sequel we always assume that $z \in X_1$, $x \in X_0$, $y \in Y_2$, and $z = x + y$. We may now write

$$g_1(z) = G(z) + \widetilde{G}(z)[y - h_2(x)] \;.$$

Define the linear homotopy

$$\begin{aligned} H(t, z) &:= (1 - t)g_1(z) + tG(z) \\ &= G(z) + (1 - t)\widetilde{G}(z)[y - h_2(x)] \;. \end{aligned}$$

We wish to show that $H \neq 0$ on $[0, 1] \times S_r X_1$ for some small $r > 0$. To achieve this recall that $g_1(x + h_2(x)) = g_2(x)$, that $g_1'(0) = P_{X_1} A|_{X_1}$, that Y_2 is invariant under A, and that $P_{Y_2} A|_{Y_2}$ is an isomorphism. Hence there is $M > 0$ such that $\|Ay\| \geq M\|y\|$ for all $y \in Y_2$. By continuity of g_1' and h_2 we may choose $r > 0$ small enough such that

$$\|P_{Y_2} \widetilde{G}(z)\| \leq \frac{M}{2}$$

if $\|z\| \leq r$, and such that g_1 has no zero in $B_r X_1$ besides 0. Fix $z \in S_r X_1$ and consider two cases: a) $y \neq h_2(x)$. Here it follows that

$$\begin{aligned} \|P_{Y_2} H(t, z)\| &\geq \|A[y - h_2(x)]\| - \|P_{Y_2} \widetilde{G}(z)[y - h_2(x)]\| \\ &\geq \frac{M}{2} \|y - h_2(x)\| > 0 \;. \end{aligned}$$

b) $y = h_2(x)$. In this case we conclude

$$\|H(t,z)\| = \|g_1(x + h_2(x))\| > 0 \ .$$

Hence the linear homotopy from g_1 to G has no zero on $S_r Z$, and $\deg_{\mathrm{loc}}(g_1, 0) = \deg(G, U_r, 0)$.

It remains to calculate $\deg(G, U_r, 0)$. Since $P_{X_0} G(z) = g_1(x + h_2(x)) = g_2(x)$ is independent of y, it is easily seen that $\deg(G, U_r, 0) = \operatorname{sign}\det(P_{Y_2} A|_{Y_2}) \deg_{\mathrm{loc}}(g_2, 0)$. This finishes the proof. □

3. The Nonlinear Superposition Principle

Let E be a real separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle_E$ and associated norm $\|\cdot\|_E$. Let $\mathcal{G}$ be an Abelian group acting isometrically on E, where we denote the group operation by $+$, inversion in $\mathcal{G}$ by $-$, and the group action on E by $a \star u$, if $a \in \mathcal{G}$ and $u \in E$. Suppose moreover that $\mathcal{G}$ is a directed set, where the direction will be denoted by $\succ$. For simplicity we adopt the terminology of saying that a *is larger than* b if $a, b \in \mathcal{G}$ and $a \succ b$. If a statement holds for all a larger that some $A \in \mathcal{G}$, then we say that the statement holds *for a large enough*. If X is a metric space and $f \colon \mathcal{G} \to X$ is a net, then by saying $f(a) \to x$ *as* $a \to \infty$ we mean that $\lim_{a \in \mathcal{G}} f(a) = x$ or, in other words, that the net limit of f is x. This convention also applies to other type of limiting processes over $\mathcal{G}$.

Recall that a sequence (a_n) in $\mathcal{G}$ is called *cofinal* if for every A in $\mathcal{G}$ there is n_0 in $\mathbb{N}$ such that $a_n \succ A$ whenever $n \geq n_0$. If $\mathcal{G}$ contains cofinal sequences then all limiting processes with respect to nets into metric spaces can be examined by only considering cofinal sequences.

We assume the following additional conditions on $\mathcal{G}$ and E:

(G3.1) $\mathcal{G}$ contains cofinal sequences.

(G3.2) If (a_n) is a cofinal sequence in $\mathcal{G}$ and $a \in \mathcal{G}$, then $(-a_n)$ and $a + a_n$ are also cofinal.

(G3.3) If (a_n) is a cofinal sequence in $\mathcal{G}$ and $u \in E$, then $a_n \star u \rightharpoonup 0$.

Note that by (G3.1) and (G3.3) $\mathcal{G}$ is infinite if E is not trivial. Let us also consider

(G3.4) Every infinite subset $\mathcal{A}$ of $\mathcal{G}$ contains a cofinal sequence.

The following definition describes one of the basic concepts for the proof of the superposition principle. It makes the statement precise that a vector field behaves asymptotically like an additive map.

3.1 Definition. If X and Y are Banach spaces and $f \colon X \to Y$ is a map, then we say that f *has the BL-splitting property*, *satisfies the BL-splitting condition*, or *BL-splits*, if for every weakly convergent sequence (x_n) in X with $x_n \rightharpoonup x$ it holds that

$$f(x_n) - f(x_n - x) \to f(x)$$

in Y, as $n \to \infty$.

3.2 Remark. The letters BL in the definition above represent the use of Brezis-Lieb type Lemmata to prove that the BL-splitting property holds.

3.3 Remark. For every $T \in \mathcal{L}(E)$ the maps $u \mapsto \langle Tu, u\rangle_E$ and $u \mapsto Tu$ have the BL-splitting property. In particular the map $u \mapsto \|u\|_E^2$ BL-splits. If f BL-splits then necessarily $f(0) = 0$.

Now consider a differentiable map $\Psi \colon E \to \mathbb{R}$ and denote its gradient by Λ. We assume the following properties for Ψ:

(F3.1) There is α in $(0, 1]$ such that $\Psi \in C^{2+\alpha}(E, \mathbb{R})$, uniformly on bounded subsets.

(F3.2) Ψ, Ψ' and Ψ'' have the BL-splitting property.

(F3.3) Λ is weakly sequentially continuous.

(F3.4) For every u in E the operator $\Lambda'(u)$ is compact.

(F3.5) Ψ is invariant under the action of $\mathcal{G}$.

We also consider $L \in \mathcal{L}(E)$ with the properties

(L3.1) L is a selfadjoint isomorphism. Its spectrum is a finite set.

(L3.2) L is equivariant under the action of $\mathcal{G}$.

Define the functional $\Phi \colon E \to \mathbb{R}$ by

$$\Phi(u) := \frac{1}{2}\langle Lu, u\rangle_E - \Psi(u) \ , \tag{3.1}$$

so Φ is also in $C^{2+\alpha}(E, \mathbb{R})$, uniformly on bounded subsets, and Φ is invariant under the action of $\mathcal{G}$. Denote the gradient of Φ by Γ. From (F3.2) and Remark 3.3 it follows that

$$\Phi \text{ and } \Phi' \text{ have the BL-splitting property.} \tag{3.2}$$

Note however that this is not true for Φ'' due to the quadratic first term in the definition of Φ.

If $\bar{u}$ is a critical point of Φ from our conditions on Ψ and L it follows that $\Gamma'(\bar{u})$ is a selfadjoint Fredholm operator of index 0. Hence $\operatorname{rdeg}_{\mathrm{loc}}(\Phi, \bar{u})$ is well defined if $\bar{u}$ is an *isolated* critical point of Φ.

To state the nonlinear superposition principle recall the definition of the set $\mathcal{K}$ of nontrivial critical points of Φ and of the sets $\mathcal{K}_c^d$ given in Section 1.3.

3.4 Theorem. a) *Suppose that A is a finite set of isolated critical points of Φ, such that Φ has nonzero reduced local degree at $\bar{u}$ for every $\bar{u}$ in A. Then A generates multibump critical points of Φ.*

b) *Suppose that* (G3.4) *holds and that $\bar{u} \neq 0$ is an isolated critical point of Φ, such that Φ has nonzero reduced local degree at $\bar{u}$. Then $\mathcal{K}_{kc-\varepsilon}^{kc+\varepsilon}/\mathcal{G}$ is infinite for $c := \Phi(\bar{u})$ and for every $\varepsilon > 0$ and every k in $\mathbb{N} \setminus \{1\}$.*

The proof of this theorem will be given in Section 3.2.

3.1. Technical Preliminaries

In this section we prepare the proof of Theorem 3.4.

3.5 Lemma. *Let X, Y, Z and P_X, P_Y be as in Definition 2.1. Suppose that U is an open neighborhood of 0 in Z and that $f, g \in C^{1+\alpha}(U, Z)$ satisfy*

$$\|f\|_{C^{1+\alpha}(U,Z)}, \|g\|_{C^{1+\alpha}(U,Z)} \leq M_0$$

for some $M_0 > 0$. Suppose moreover that there is $M_1 > 0$ such that

$$\|f'(0)y\| \geq M_1\|y\|$$

for $y \in Y$. Then there is $r_0 > 0$, only depending on M_0, M_1 and α, such that for every $r \in (0, r_0]$ with $B_r(0) \subseteq U$ there are $\varepsilon, \delta > 0$, only depending on r, M_0, M_1, and α, with the following property: If

$$\max\{\, \|f(0) - g(0)\|, \|f'(0) - g'(0)\| \,\} \leq \varepsilon$$

then

$$\|f(z) - g(z)\| < \|f(z) - f(0)\|$$

for every $z \in S_r Z$ with $\|P_X z\| \leq \delta$.

Proof. Set $\mu := 2M_0/M_1$. In the sequel we always assume that $x \in X$, $y \in Y$, $z = x + y$ and

$$\|x\| \leq \frac{1}{1+\mu}\|z\| \, . \tag{3.3}$$

It then holds that

$$\mu\|x\| \leq \|y\|$$

and

$$\|z\| \leq \|x\| + \|y\| \leq \left(1 + \frac{1}{\mu}\right)\|y\| \, .$$

From these inequalities it follows that

$$\begin{aligned} \|f'(0)z\| &\geq \|f'(0)y\| - \|f'(0)x\| \geq M_1\|y\| - M_0\|x\| \\ &\geq \left(M_1 - \frac{M_0}{\mu}\right)\|y\| = \frac{M_1}{2}\|y\| \geq C_1\|z\| \end{aligned} \tag{3.4}$$

with

$$C_1 := \frac{\mu M_1}{2(1+\mu)} \, .$$

Moreover, from the bounds on f we find

$$\|f(z) - f(0) - f'(0)z\| = \left\|\int_0^1 (f'(sz) - f'(0))z \,\mathrm{d}s\right\| \leq \frac{M_0}{1+\alpha}\|z\|^{1+\alpha} \, . \tag{3.5}$$

Together with (3.4) we obtain

$$\|f(z) - f(0)\| \geq \|f'(0)z\| - \|f(z) - f(0) - f'(0)z\| \geq \left(C_1 - \frac{M_0}{1+\alpha}\|z\|^\alpha\right)\|z\| \ . \tag{3.6}$$

Now set $h := f - g$. Then $\|h\|_{C^{1+\alpha}(U,Z)} \leq 2M_0$ and as in (3.5)

$$\|h(z)\| \leq \|h(0)\| + \|h'(0)\| \cdot \|z\| + \frac{2M_0}{1+\alpha}\|z\|^{1+\alpha} \ . \tag{3.7}$$

From (3.3), (3.6), and (3.7) it follows that if $r > 0$ is small enough we may choose $\varepsilon > 0$ small enough and define $\delta := r/(1+\mu)$ such that $\|z\| = r$, $\max\{\, \|h(0)\|, \|h'(0)\| \,\} \leq \varepsilon$ and $\|x\| \leq \delta$ implies $\|h(z)\| < \|f(z) - f(0)\|$. The proof is finished. □

3.6 Lemma. *Suppose that Z is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, that X_1 and X_2 are nontrivial closed subspaces, and that*

$$\gamma := \sup\{\, |\langle x_1, x_2 \rangle| \mid x_i \in S_1 X_i \,\} < 1 \ .$$

Then trivially $X_1 \cap X_2 = \{0\}$. Set $X := X_1 \oplus X_2$. If we denote, for $i = 1, 2$, by P_i and P the orthogonal projections in Z onto X_i and X, respectively, then $(P_1 + P_2)|_X$ is an isomorphism of X and

$$\|P - P_1 - P_2\| \leq \frac{2\gamma}{1-\gamma} \ .$$

Proof. Define $Q_i := P_i|_X$ and $Y_i := \mathcal{N}(P_i) \cap X = \mathcal{N}(Q_i)$. If $x \in S_1 X_2$ and $y = Q_1 x$ then

$$y = \left\langle x, \frac{y}{\|y\|} \right\rangle \frac{y}{\|y\|}$$

and hence $\|y\| \leq \gamma$. This implies $\|Q_1|_{X_2}\| \leq \gamma$ and $\|Q_1 Q_2\| \leq \gamma$. We have $\mathcal{N}(I - Q_1) \cap X_2 = X_1 \cap X_2 = \{0\}$. If $y \in Y_1$ and $y = x_1 + x_2$ such that $x_i \in X_i$, then

$$(I - Q_1)x_2 = (I - Q_1)[x_1 + x_2] = (I - Q_1)y = y \ .$$

These facts show that $(I - Q_1)|_{X_2}$ is an isomorphism from X_2 onto Y_1. From [26, Thm. I.6.34] it now follows that

$$\|I_X - Q_1 - Q_2\| \leq \gamma < 1 \tag{3.8}$$

and hence that $Q_1 + Q_2 = (P_1 + P_2)|_X$ is an isomorphism. It is plain that $P = (Q_1 + Q_2)^{-1}(P_1 + P_2)$. Moreover by (3.8)

$$\begin{aligned} \|P - P_1 - P_2\| &= \|(I_X - (Q_1 + Q_2)^{-1})(P_1 + P_2)\| \\ &\leq 2\left\| I_X - \sum_{k=0}^{\infty} (I_X - Q_1 - Q_2)^k \right\| \leq \frac{2\gamma}{1-\gamma} \ . \end{aligned}$$

□

3.2. Proof of Theorem 3.4

In this section we will write $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_E$ and $\|\cdot\| := \|\cdot\|_E$. Moreover, for every closed subspace X of E we denote by P_X the orthogonal projection in E onto X.

We restrict ourselves to proving that the set $\{\bar{u}\}$ generates critical points with two bumps if $\bar{u}$ is an isolated critical point of Φ with nonzero reduced local degree. The general result can be obtained by making straightforward modifications to the proof below.

Let us first state some useful facts. Here we write $\Sigma(a)$ for the isometry that corresponds to a in $\mathcal{G}$. Suppose that X is a closed subspace of E and that $a \in \mathcal{G}$. Since $\mathcal{G}$ acts isometrically on E we have

$$(\Sigma(a)X)^{\perp} = \Sigma(a)X^{\perp} \; . \tag{3.9}$$

From (3.9) it follows that

$$P_{\Sigma(a)X}\Sigma(a) = \Sigma(a)P_X \; . \tag{3.10}$$

We introduce additional notation for convenience. Recall that Λ is the gradient of Ψ and Γ is the gradient of Φ. Denote $u_a := \bar{u} + a \star \bar{u}$, $K := \Lambda'(\bar{u})$, $K_a := \Lambda'(a \star \bar{u})$, $R := L - K$, and $R_a := L - K_a$. Since Λ is equivariant by (F3.5), it follows that

$$\begin{aligned} K_a\Sigma(a) &= \Sigma(a)K \\ R_a\Sigma(a) &= \Sigma(a)R \; . \end{aligned} \tag{3.11}$$

By (F3.4) K is compact and selfadjoint.

By (G3.3) $u_a \rightharpoonup \bar{u}$ as $a \to \infty$ so that (F3.2) and (3.2) imply

$$\begin{aligned} \Phi(u_a) - \Phi(a \star \bar{u}) - \Phi(\bar{u}) &\to 0 \\ \Gamma(u_a) - \Gamma(a \star \bar{u}) - \Gamma(\bar{u}) &\to 0 \\ \Lambda'(u_a) - \Lambda'(a \star \bar{u}) - \Lambda'(\bar{u}) &\to 0 \end{aligned} \tag{3.12}$$

as $a \to \infty$. By invariance

$$\Phi(a \star \bar{u}) = \Phi(\bar{u})$$

for all a in $\mathcal{G}$, so

$$\Phi(u_a) \to 2\Phi(\bar{u}) \qquad \text{as } a \to \infty. \tag{3.13}$$

By equivariance $\Gamma(a \star \bar{u}) = a \star \Gamma(\bar{u}) = 0$ for all a in $\mathcal{G}$, so

$$\Gamma(u_a) \to 0 \qquad \text{as } a \to \infty. \tag{3.14}$$

Moreover, from (3.12) we obtain

$$\Gamma'(u_a) = L - K - K_a + o(1) \quad \text{as } a \to \infty. \tag{3.15}$$

We start by proving item a) of Theorem 3.4. To highlight the basic idea of the proof we first assume that $\bar{u}$ is a nondegenerate critical point of Φ. This case is considerably simpler to treat. Since by (3.14) u_a is an approximate zero of Γ for large a we can apply Corollary 2.6 if we can show that $\Gamma'(u_a)$ is an isomorphism for large a and that $\|\Gamma'(u_a)^{-1}\|$ remains bounded as $a \to \infty$. Therefore set $M := \|R^{-1}\|$. We claim that

$$\liminf_{a\in\mathcal{G}} \inf_{y\in S_1E} \|\Gamma'(u_a)y\| \geq \frac{1}{M} \,. \tag{3.16}$$

To show this consider a cofinal sequence (a_m) in $\mathcal{G}$ and a sequence (y_m) in S_1E. Extracting subsequences we may assume that

$$\begin{aligned} y_m &\rightharpoonup v \\ (-a_m)\star y_m &\rightharpoonup w \,. \end{aligned}$$

We set $z_m := y_m - v - a_m \star w$ so by (G3.2) and (G3.3)

$$\begin{aligned} z_m &\rightharpoonup 0 \\ (-a_m)\star z_m &\rightharpoonup 0 \,. \end{aligned} \tag{3.17}$$

Since K is compact, from these facts and (3.11) we obtain

$$\begin{aligned} K[a_m \star w] &\to 0 \\ K_{a_m} v = \Sigma(a_m) K[(-a_m)\star v] &\to 0 \\ K_{a_m} z_m = \Sigma(a_m) K[(-a_m)\star z_m] &\to 0 \end{aligned}$$

and hence

$$(L - K - K_{a_m}) y_m = Rv + \Sigma(a_m)Rw + Rz_m + o(1) \tag{3.18}$$

as $m \to \infty$. Using (3.15), (3.17), (3.18) and Remark 3.3 we obtain

$$\begin{aligned} \|\Gamma'(u_{a_m})y_m\|^2 &= \|(L-K-K_{a_m})y_m\|^2 + o(1) \\ &= \|Rv\|^2 + \|Rw\|^2 + \|Rz_m\|^2 + o(1) \\ &\geq \frac{1}{M^2}(\|v\|^2 + \|w\|^2 + \|z_m\|^2) + o(1) \\ &= \frac{1}{M^2}\|y_m\|^2 + o(1) \,. \end{aligned}$$

Since (a_m) and (y_m) were chosen arbitrarily, (3.16) is proved.

By (3.16) and the selfadjointness of $\Gamma'(u_a)$ we may pick A in $\mathcal{G}$ such that $\Gamma'(u_a)$ is invertible with

$$\|\Gamma'(u_a)^{-1}\| \leq 2M$$

for every $a \succ A$. Choosing A larger if necessary, this fact together with (F3.1), (3.14) and Corollary 2.6 yields a constant C_2, independent of a, such that Γ has a zero in $B(C_2\Gamma(u_a), u_a)$ for every $a \succ A$. Therefore, for a large enough Γ has a zero v in $B_\varepsilon(u_a)$

such that $|\Phi(v) - \Phi(u_a)| \leq \varepsilon/2$. Here we have used (F3.1) again. The number ε is taken from the statement of the theorem. If a is chosen large enough then also $|\Phi(u_a) - 2\Phi(\bar{u})| \leq \varepsilon/2$ by (3.13), so $|\Phi(v) - 2\Phi(\bar{u})| \leq \varepsilon$. Hence we have proved the existence of two-bump critical points of Φ near the sum of translates of the nondegenerate critical point $\bar{u}$.

Now we take up the proof in the case that $\bar{u}$ is degenerate. Property (L3.1), the compactness and selfadjointness of K, and the separability of E imply the existence of a sequence $(X_n)_{n\in\mathbb{N}_0}$ of finite-dimensional R-invariant subspaces of E such that

$$\mathcal{N}(R) = X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

and

$$E = \overline{\bigcup_{n=0}^{\infty} X_n} \; . \tag{3.19}$$

Moreover, the spaces $Y_n := X_n^{\perp}$ are also invariant under R. We set $X_{a,n} := X_n + \Sigma(a)X_n$ and $Y_{a,n} := (X_{a,n})^{\perp} = Y_n \cap (\Sigma(a)Y_n)$.

3.7 Remark. If (u_m) is a bounded sequence in E such that $P_{X_n}u_m \to 0$ as $m \to \infty$, for all n in $\mathbb{N}_0$, then $u_m \rightharpoonup 0$ as $m \to \infty$. To see this fix some v in E and n in $\mathbb{N}_0$. Using that $\|u_m\| \leq C$ for some positive constant C and all m we obtain

$$\limsup_{m\to\infty} |\langle u_m, v\rangle| \leq \limsup_{m\to\infty} |\langle P_{X_n}u_m, v\rangle| + \limsup_{m\to\infty} |\langle u_m, P_{Y_n}v\rangle| \leq C\|P_{Y_n}v\| \; .$$

Letting $n \to \infty$, the claim follows from (3.19).

Define

$$\gamma(a,n) := \sup\{\, |\langle x_1, x_2\rangle| \mid x_1, (-a)\star x_2 \in S_1X_n \,\} \; .$$

Since X_n is finite-dimensional it follows that

$$\gamma(a,n) \to 0 \tag{3.20}$$

as $a \to \infty$, for n fixed. Consider the constant $M := \|(P_{Y_0}R|_{Y_0})^{-1}\|$. Since Y_n is R-invariant for $n \in \mathbb{N}_0$ it follows that

$$\|(P_{Y_n}R|_{Y_n})^{-1}\| \leq M \tag{3.21}$$

for $n \in \mathbb{N}_0$. We have the following asymptotic properties:

3.8 Lemma. *For fixed n in $\mathbb{N}_0$ it holds that:*

(i) $\lim_{a\in\mathcal{G}} \|P_{X_n}|_{\Sigma(a)X_n}\| = \lim_{a\in\mathcal{G}} \|P_{\Sigma(a)X_n}|_{X_n}\| = 0$

(ii) $\lim_{a\in\mathcal{G}} \|P_{X_n}\Gamma'(u_a)|_{\Sigma(a)X_n}\| = \lim_{a\in\mathcal{G}} \|P_{\Sigma(a)X_n}\Gamma'(u_a)|_{X_n}\| = 0$

(iii) $\lim_{a\in\mathcal{G}} \|P_{X_n}(\Gamma'(u_a) - R)|_{X_n}\| = \lim_{a\in\mathcal{G}} \|P_{\Sigma(a)X_n}(\Gamma'(u_a) - R_a)|_{\Sigma(a)X_n}\| = 0$

(iv) $\lim_{a\in\mathcal{G}} \|P_{X_{a,n}}\Gamma'(u_a)|_{Y_{a,n}}\| = 0$

(v) $\liminf_{a\in\mathcal{G}} \inf_{y\in S_1 Y_{a,n}} \|P_{Y_{a,n}}\Gamma'(u_a)y\| \geq 1/M$

The proof of this lemma will be given at the end of this section.

For every a in $\mathcal{G}$ the operator $P_{Y_{a,n}}\Gamma'(u_a)|_{Y_{a,n}} \in \mathcal{L}(Y_{a,n})$ is selfadjoint. Hence from Lemma 3.8(v) it follows that for large a it is invertible with

$$\|(P_{Y_{a,n}}\Gamma'(u_a)|_{Y_{a,n}})^{-1}\| \leq 2M .$$

Combining this fact with (F3.1), (3.14), (3.21) and the fact that the norms of orthogonal projections are bounded by 1, we can apply Lemma 2.4 and Remark 2.5 to obtain constants $0 < r_1 \leq r_2$ and $C_2 > 0$, independently of n, and for each n in $\mathbb{N}_0$ some $A(n)$ in $\mathcal{G}$ such that for $a \succ A(n)$ the following holds:

- The reduction F_n of Γ at $\bar{u}$ to X_n exists on $B_{r_1}X_n$. It comes with a map $\kappa_n\colon B_{r_1}X_n \to B_{r_2}Y_n$ such that if $x \in B_{r_1}X_n$ and $y \in B_{r_2}Y_n$ then $P_{Y_n}\Gamma(\bar{u}+x+y) = 0$ if and only if $y = \kappa_n(x)$. The following properties hold:

$$\begin{aligned} \|F_n\|_{C^{1+\alpha}(B_{r_1}X_n, X_n)} &\leq C_2 \\ \|\kappa_n\|_{C^{1+\alpha}(B_{r_1}X_n, B_{r_2}Y_n)} &\leq C_2 \\ F_n(x) &= \Gamma(\bar{u} + x + \kappa_n(x)) \end{aligned} \tag{3.22}$$

  and

  (3.23) The zeros of F_n in $B_{r_1}X_n$ are in one-to-one correspondence with the zeros of Γ in $u_a + B_{r_1}X_n + B_{r_2}Y_n$.

- The reduction $G_{a,n}$ of Γ at u_a to $X_{a,n}$ exists on $B_{r_1}X_{a,n}$. It comes with a map $\eta_{a,n}\colon B_{r_1}X_{a,n} \to B_{r_2}Y_{a,n}$ such that if $x \in B_{r_1}X_{a,n}$ and $y \in B_{r_2}Y_{a,n}$ then $P_{Y_{a,n}}\Gamma(u_a + x + y) = 0$ if and only if $y = \eta_{a,n}(x)$. The following properties hold:

$$\begin{aligned} \|G_{a,n}\|_{C^{1+\alpha}(B_{r_1}X_{a,n}, X_{a,n})} &\leq C_2 \\ \|\eta_{a,n}\|_{C^{1+\alpha}(B_{r_1}X_{a,n}, B_{r_2}Y_{a,n})} &\leq C_2 \\ G_{a,n}(x) &= \Gamma(u_a + x + \eta_{a,n}(x)) \end{aligned} \tag{3.24}$$

  and

  (3.25) The zeros of $G_{a,n}$ in $B_{r_1}X_{a,n}$ are in one-to-one correspondence with the zeros of Γ in $u_a + B_{r_1}X_{a,n} + B_{r_2}Y_{a,n}$.

By (3.14) and Lemma 2.4(iii) we can take r_1 and r_2 as small as we wish, as long as we choose $A(n)$ large enough for every n. By (F3.1) and (3.23) we may thus assume that r_1 and r_2 are chosen such that if $n \in \mathbb{N}_0$ and $a \in \mathcal{G}$ with $a \succ A(n)$, then

$$0 \text{ is the only zero of } F_n \text{ in } B_{r_1}X_n \tag{3.26}$$

$$r_1 + r_2 \leq \varepsilon \tag{3.27}$$

and if $u \in u_a + B_{r_1}X_{a,n} + B_{r_2}Y_{a,n}$ then

$$|\Phi(u) - \Phi(u_a)| \le \varepsilon/2 \,, \tag{3.28}$$

where ε is from the statement of the theorem. Subsequently we will enlarge each $A(n)$ even more, in finitely many steps, to ensure that certain additional conditions are met.

Define for every n in $\mathbb{N}_0$ the Banach space $Z_n := X_n \times X_n$ with norm

$$\|(x^1, x^2)\|_{Z_n} = \|x^1\| + \|x^2\| \,.$$

Also define

$$\begin{aligned} f_n \colon B_{r_1}X_n \times B_{r_1}X_n &\to Z_n \\ (x^1, x^2) &\mapsto (F_n(x^1), F_n(x^2)) \end{aligned}$$

and for $a \succ A(n)$

$$\begin{aligned} g_{a,n} \colon B_{r_1/2}X_n \times B_{r_1/2}X_n &\to Z_n \\ (x^1, x^2) &\mapsto (P_{X_n}G_{a,n}(x^1 + a \star x^2), \Sigma(-a)P_{\Sigma(a)X_n}G_{a,n}(x^1 + a \star x^2)) \,. \end{aligned}$$

By (3.20) and Lemma 3.6 $(P_{X_n} + P_{\Sigma(a)X_n})|_{X_{a,n}}$ is invertible if $a \succ A(n)$ for $A(n)$ chosen large enough. Now (3.25) yields that for $a \succ A(n)$:

(3.29) The zeros of $g_{a,n}$ are in one-to-one correspondence with the zeros of Γ in $u_a + B_{r_1}X_{a,n} + B_{r_2}Y_{a,n}$.

Note that

$$0 \text{ is the only zero of } f_n \tag{3.30}$$

by (3.26). From Lemma 2.7 we obtain

$$\deg_{\text{loc}}(f_n, 0) = \deg_{\text{loc}}(F_n, 0)^2 = \deg_{\text{loc}}(F_0, 0)^2 = \text{rdeg}_{\text{loc}}(\Gamma, \bar{u})^2 \neq 0 \,. \tag{3.31}$$

Therefore our goal in the rest of the proof is to show that $g_{a,n}$ approximates f_n well enough for appropriate a and n such that by homotopy invariance of the degree we can conclude.

Let us consider n fixed for the moment. Since $\Gamma(\bar{u}) = 0$ and since X_n and Y_n are invariant under $R = \Gamma'(\bar{u})$, Lemma 2.4(vi) yields

$$f_n'(0) = \begin{pmatrix} R|_{X_n} & 0 \\ 0 & R|_{X_n} \end{pmatrix} \,.$$

It follows that

$$V_n := \mathcal{N}(f_n'(0)) = X_0 \times X_0 \tag{3.32}$$

$$W_n := \mathcal{R}(f_n'(0)) = (Y_0 \cap X_n) \times (Y_0 \cap X_n) \tag{3.33}$$

$$Z_n = V_n \oplus W_n \tag{3.34}$$

and

$$V_n \text{ and } W_n \text{ are invariant under } f_n'(0).$$

If $(y^1, y^2) \in W_n$ then

$$\begin{aligned}(3.35)\quad \|f_n'(0)(y^1,y^2)\|_{Z_n} &= \|(Ry^1, Ry^2)\|_{Z_n} = \|Ry^1\| + \|Ry^2\| \\ &\geq \frac{1}{M}(\|y^1\| + \|y^2\|) = \frac{1}{M}\|(y^1,y^2)\|_{Z_n}\end{aligned}$$

since $y^1, y^2 \in Y_0$. Moreover

$$\|g_{a,n}(0)\|_{Z_n} \leq C\|G_{a,n}(0)\| \leq C\|\Gamma(u_a)\|$$

by Lemma 2.4(v), where C is independent of $a \succ A(n)$. Therefore from (3.14) it follows that $\|g_{a,n}(0)\|_{Z_n} \to 0$ and hence by (3.30)

$$g_{a,n}(0) = f_n(0) + o(1) \quad \text{as } a \to \infty. \tag{3.36}$$

Combining (3.14) again with Lemma 3.8(iv) and Lemma 2.4(vi) yields

$$\|G_{a,n}'(0) - P_{X_{a,n}}\Gamma'(u_a)|_{X_{a,n}}\| \to 0 \quad \text{as } a \to \infty.$$

From this fact and from Lemma 3.8(ii) and (iii) we obtain

$$\begin{aligned} g_{a,n}'(0) &= \begin{pmatrix} P_{X_n}G_{a,n}'(0)|_{X_n} & P_{X_n}G_{a,n}'(0)|_{\Sigma(a)X_n}\Sigma(a)|_{X_n} \\ \Sigma(-a)P_{\Sigma(a)X_n}G_{a,n}'(0)|_{X_n} & \Sigma(-a)P_{\Sigma(a)X_n}G_{a,n}'(0)|_{\Sigma(a)X_n}\Sigma(a)|_{X_n} \end{pmatrix} \\ &= \begin{pmatrix} P_{X_n}\Gamma'(u_a)|_{X_n} & P_{X_n}\Gamma'(u_a)|_{\Sigma(a)X_n}\Sigma(a)|_{X_n} \\ \Sigma(-a)P_{\Sigma(a)X_n}\Gamma'(u_a)|_{X_n} & \Sigma(-a)P_{\Sigma(a)X_n}\Gamma'(u_a)|_{\Sigma(a)X_n}\Sigma(a)|_{X_n} \end{pmatrix} + o(1) \\ &= \begin{pmatrix} P_{X_n}R|_{X_n} & 0 \\ 0 & \Sigma(-a)P_{\Sigma(a)X_n}R_a\Sigma(a)|_{X_n} \end{pmatrix} + o(1) \end{aligned}$$

as $a \to \infty$. Note that $P_{X_n}R|_{X_n} = R|_{X_n}$ by invariance and that

$$\Sigma(-a)P_{\Sigma(a)X_n}R_a\Sigma(a)|_{X_n} = P_{X_n}\Sigma(-a)R_a\Sigma(a)|_{X_n} = P_{X_n}R\Sigma(-a)\Sigma(a)|_{X_n} = R|_{X_n}\ .$$

Hence we arrive at

$$g_{a,n}'(0) = f_n'(0) + o(1) \quad \text{as } a \to \infty. \tag{3.37}$$

Denote by Q_{V_n} and Q_{W_n} the projections with ranges V_n and W_n respectively, defined in Z_n corresponding to the splitting (3.34). We use Lemma 3.5, (3.30), (3.35), (3.36) and (3.37) to make $A(n)$ large enough and to find $r_3 \in (0, r_1/2]$ and $\delta > 0$ with the following property:

$$a \succ A(n),\ z \in S_{r_3}Z_n,\ \|Q_{V_n}z\|_{Z_n} \leq \delta \quad \Rightarrow \quad \|f_n(z) - g_{a,n}(z)\|_{Z_n} < \|f_n(z)\|_{Z_n}\ . \tag{3.38}$$

Here we choose the constants r_3 and δ given by Lemma 3.5 independently of n and $a \succ A(n)$. This is possible since by (3.22) and (3.24) there are independent bounds on $\|f_n\|_{C^{1+\alpha}}$ and $\|g_{a,n}\|_{C^{1+\alpha}}$, and since (3.35) is independent of a and n.

Further enlarging $A(n)$ we may assume by Lemma 3.8(i) that for $a \succ A(n)$

$$\|P_{X_n}|_{\Sigma(a)X_n}\| \le \frac{1}{n}\,, \qquad \|P_{\Sigma(a)X_n}|_{X_n}\| \le \frac{1}{n}\,. \tag{3.39}$$

Now we explicitly consider the dependency of the above statements on n again. We claim that there are n_0 in $\mathbb{N}_0$ and $\tilde{A} \succ A(n_0)$ in $\mathcal{G}$ such that the following implication holds:

$$a \succ \tilde{A},\ \beta \in \mathbb{R},\ z \in S_{r_3} Z_{n_0},\ \|Q_{V_{n_0}} z\|_{Z_{n_0}} \ge \delta,\ g_{a,n_0}(z) = \beta f_{n_0}(z) \quad \Rightarrow \quad \beta > 0\,. \tag{3.40}$$

To prove the claim we argue by contradiction. If the claim is false, by a diagonal selection process there exist a cofinal sequence (a_n) in $\mathcal{G}$, a sequence (β_n) in $\mathbb{R}$, and a sequence (z_n) in E with the following properties:

$$a_n \succ A(n) \tag{3.41}$$

$$z_n \in S_{r_3} Z_n \tag{3.42}$$

$$\beta_n \le 0 \tag{3.43}$$

$$\|Q_{V_n} z_n\| \ge \delta \tag{3.44}$$

$$g_{a_n,n}(z_n) = \beta_n f_n(z_n)\,. \tag{3.45}$$

Set $z_n := (z_n^1, z_n^2)$ where $z_n^i \in X_n$ for $i = 1, 2$. Denote $x_n^i := P_{X_0} z_n^i$ and $x_n := (x_n^1, x_n^2)$. Then $x_n = Q_{V_n} z_n \in V_n$ and $\delta \le \|x_n\|_{Z_n} = \|x_n^1\| + \|x_n^2\|$. After extraction of a subsequence and relabeling we may assume that $\|x_n^1\| \ge \delta/2$ for all n (otherwise exchange the roles of x_n^1 and x_n^2 below). Since $\|z_n^1\| \le \|z_n\|_{Z_n} = r_3$, after repeatedly passing to a subsequence we may assume that $z_n^1 \rightharpoonup z^1 \in E$. Since P_{X_0} is finite-dimensional, $x_n^1 \to P_{X_0} z^1$. This yields $\|P_{X_0} z^1\| \ge \delta/2$ and hence $z^1 \ne 0$.

We have to consider the maps κ_n and $\eta_{a,n}$ obtained in the definition of the reductions F_n and $G_{a,n}$. By (3.22) $\|\kappa_n(z_n^1)\|$ remains bounded. Since $\kappa_n(z_n^1) \in Y_n$ for all n it follows that $\lim_{n\to\infty} \|P_{X_m} \kappa_n(z_n^1)\| = 0$ for every m in $\mathbb{N}_0$. Remark 3.7 yields

$$\kappa_n(z_n^1) \rightharpoonup 0 \qquad \text{as } n \to \infty,$$

and from (F3.3) we obtain

$$F_n(z_n^1) = \Gamma(\bar{u} + z_n^1 + \kappa_n(z_n^1)) \rightharpoonup \Gamma(\bar{u} + z^1)\,. \tag{3.46}$$

Let us turn to the weak limit of $G_{a_n,n}(z_n^1 + a_n \star z_n^2)$. Since (a_n) is cofinal we have

$$u_{a_n} \rightharpoonup \bar{u}\,. \tag{3.47}$$

Moreover $\|a_n \star z_n^2\| \le r_3$ and $(a_n \star z_n^2) \in \Sigma(a_n) X_n$. Hence $\|P_{X_n}[a_n \star z_n^2]\| \le r_3/n$ by (3.39). Therefore $\|P_{X_m}[a_n \star z_n^2]\| \to 0$ as $n \to \infty$, for all m in $\mathbb{N}_0$. Again by Remark 3.7

$$a_n \star z_n^2 \rightharpoonup 0 \quad \text{as } n \to \infty. \tag{3.48}$$

Since $\eta_{a_n,n}(z_n^1 + a_n \star z_n^2) \in Y_{a_n,n} \subseteq Y_n$ is bounded by (3.24), Remark 3.7 yields

$$\eta_{a_n,n}(z_n^1 + a_n \star z_n^2) \rightharpoonup 0 \quad \text{as } n \to \infty. \tag{3.49}$$

From (3.47), (3.48), and (3.49) we obtain

$$\begin{aligned} &P_{X_n} G_{a_n,n}(z_n^1 + a_n \star z_n^2) \\ &\qquad = P_{X_n}\Gamma(u_{a_n} + z_n^1 + a_n \star z_n^2 + \eta_{a_n,n}(z_n^1 + a_n \star z_n^2)) \rightharpoonup \Gamma(\bar{u} + z^1) \ . \end{aligned} \tag{3.50}$$

Here we have used the fact that if $u_n \rightharpoonup u$ in E, then also $P_{X_n} u_n \rightharpoonup u$ in E by (3.19).

Recall that $\bar{u}$ is the only zero of Γ in $B_{r_3}(\bar{u}; E)$, by (3.23) and (3.26). Therefore $z^1 \neq 0$ and $\|z^1\| \leq r_3$ imply that

$$\Gamma(\bar{u} + z^1) \neq 0 \ . \tag{3.51}$$

From (3.45) and the definition of f_n and $g_{a,n}$ it follows that

$$\langle P_{X_n} G_{a_n,n}(z_n^1 + a_n \star z_n^2), \Gamma(\bar{u} + z^1)\rangle = \beta_n \langle F_n(z_n^1), \Gamma(\bar{u} + z^1)\rangle \ .$$

Combining this with (3.46), (3.50), and (3.51) yields $\beta_n \to 1$ as $n \to \infty$, in contradiction with (3.43). This concludes the proof of the claim and of (3.40).

We are now in the position to finish the proof of the theorem. Fix a in $\mathcal{G}$ with $a \succ \tilde{A} \succ A(n_0)$ such that

$$|\Phi(u_a) - 2\Phi(\bar{u})| \leq \frac{\varepsilon}{2} \ . \tag{3.52}$$

This is possible by (3.13). From (3.38) and (3.40) we deduce the implication

$$z \in S_{r_3} Z_{n_0},\ \beta \in \mathbb{R},\ g_{a,n_0}(z) = \beta f_{n_0}(z) \quad \Rightarrow \quad \beta > 0 \ .$$

Together with (3.30) this implies that the linear homotopy $H(t,z) := (1-t) f_{n_0}(z) + t g_{a,n_0}(z)$, defined on $[0,1] \times B_{r_3} Z_{n_0}$, satisfies

$$0 \notin H([0,1] \times S_{r_3} Z_{n_0}) \ .$$

In view of (3.31) we have $\deg(f_{n_0}, U_{r_3} Z_{n_0}, 0) \neq 0$. By the homotopy invariance of the degree also g_{a,n_0} must have a zero in $U_{r_3} Z_{n_0}$. Hence (3.29) yields a zero v of Γ in $u_a + B_{r_1} X_{a,n} + B_{r_2} Y_{a,n}$. From (3.27), (3.28), and (3.52) we now deduce

$$\|u_a - v\| \leq \varepsilon$$

and

$$|\Phi(v) - 2\Phi(\bar{u})| \leq \varepsilon \ .$$

This proves the first assertion of the theorem.

To show item b) of Theorem 3.4 assume that (G3.4) holds. Following [15], for $n \in \mathbb{N}$ and for a subset $W \subseteq E$ let us denote

$$\mathcal{T}_n(W) := \left\{ \sum_{i=1}^{k} a_i \star u_i \;\middle|\; 1 \le k \le n,\ u_i \in W,\ a_i \in \mathcal{G} \right\} .$$

Using (G3.4) it can be proved in the same way as in [15, Prop. 1.55] that

$$\delta_n(W) := \inf\{\, \|u - v\| \mid u, v \in \mathcal{T}_n(W),\ u \ne v \,\} > 0 \qquad \text{if } W \text{ is finite.} \tag{3.53}$$

In the proof one only needs to replace "bounded sequence" with "sequence with no cofinal subsequence" and "unbounded sequence" with "sequence that contains cofinal subsequences".

Fix $\varepsilon > 0$ and $k \in \mathbb{N} \setminus \{1\}$. Set $\delta := \delta_k(\{\bar{u}\})$ as in (3.53). By what we have already proved there is A in $\mathcal{G}$ such that, denoting

$$X := \left\{ \sum_{i=1}^{k} a_i \star \bar{u} \;\middle|\; a_i \in \mathcal{G},\ a_i - a_j \succ A \text{ for } i \ne j \right\} \subseteq \mathcal{T}_k(\{\bar{u}\}),$$

for every u in X it holds that

$$B_{\delta/3}(u) \cap \mathcal{K}_{kc-\varepsilon}^{kc+\varepsilon} \ne \varnothing.$$

By the definition of δ it now suffices to show that $X/\mathcal{G}$ is infinite.

For this purpose, fix elements $a_1, a_2, \ldots, a_{k-1} \in \mathcal{G}$ such that $a_i - a_j \succ A$ for $i \ne j$ and

$$v := \sum_{i=1}^{k-1} a_i \star \bar{u} \ne 0 \ .$$

This is possible since $\bar{u} \ne 0$, $\|\cdot\|^2$ BL-splits and $\mathcal{G}$ contains cofinal sequences.

Let (b_n) denote a cofinal sequence in $\mathcal{G}$ such that $b_n - a_i \succ A$ and $a_i - b_n \succ A$ for all $i = 1, 2, \ldots, k-1$ and all n. It follows that $v + b_n \star \bar{u} \in X$ for all n. Now we argue by contradiction. If $X/\mathcal{G}$ is finite, after passing to a subsequence there is a sequence (c_n) in $\mathcal{G}$ and some $w \in E$ such that $v + b_n \star \bar{u} = c_n \star w$ for all n. Clearly, (c_n) cannot contain a constant subsequence, since $\bar{u} \ne 0$ and (b_n) is cofinal. Passing to a subsequence, by (G3.4) we may therefore assume that (c_n) is cofinal. Then $v = c_n \star w - b_n \star \bar{u} \rightharpoonup 0$ as $n \to \infty$. This contradicts $v \ne 0$.

The proof of the theorem is complete. □

Proof of Lemma 3.8. (i) is a direct consequence of (3.20). To prove the other parts we first show

$$\lim_{a\in\mathcal{G}} \|P_{X_n} K_a\| = \lim_{a\in\mathcal{G}} \|P_{\Sigma(a)X_n} K\| = 0 \tag{3.54}$$

$$\lim_{a\in\mathcal{G}} \|K_a|_{X_n}\| = \lim_{a\in\mathcal{G}} \|K|_{\Sigma(a)X_n}\| = 0 \tag{3.55}$$

$$\lim_{a\in\mathcal{G}} \|P_{\Sigma(a)X_n} L|_{X_n}\| = \lim_{a\in\mathcal{G}} \|P_{X_n} L|_{\Sigma(a)X_n}\| = 0 \ . \tag{3.56}$$

Recall the identities given in (3.10) and (3.11). In what follows let (a_m) be any cofinal sequence in $\mathcal{G}$. Suppose that (x_m) is a sequence in S_1E. After extraction of a subsequence we may assume that there is x in E such that $(-a_m) \star x_m \rightharpoonup x$. Then $K_{a_m}x_m = \Sigma(a_m)K[(-a_m) \star x_m] \rightharpoonup 0$ since $K[(-a_m) \star x_m] \to Kx$ by the compactness of K, and by (G3.2) and (G3.3). Since P_{X_n} is finite-dimensional we obtain $P_{X_n}K_{a_m}x_m \to 0$ as $n \to \infty$. The same argument applied to $P_{\Sigma(a)X_n}K$ then yields (3.54). The proof of (3.55) is similar.

Suppose now that (x_m) is a sequence in S_1X_n. Since X_n is finite-dimensional $(-a_m) \star x_m \rightharpoonup 0$. Hence also $L[(-a_m) \star x_m] \rightharpoonup 0$. Condition (L3.2) and the compactness of P_{X_n} yield $P_{\Sigma(a)X_n}Lx_m = \Sigma(a_m)P_{X_n}L[(-a_m) \star x_m] \to 0$ as $m \to \infty$. The other half of (3.56) is proved similarly.

The statements (ii) and (iii) now follow from (3.15) and (3.54)–(3.56).

Recall that by (3.9) and the definition of X_n the subspaces $\Sigma(a)X_n$ and $\Sigma(a)Y_n$ are mutually orthogonal and invariant under R_a, if $a \in \mathcal{G}$. Therefore (3.15) and (3.54) yield

$$\lim_{a\in\mathcal{G}} \|P_{X_n}\Gamma'(u_a)|_{Y_n}\| = \lim_{a\in\mathcal{G}} \|P_{X_n}(L-K-K_a)|_{Y_n}\| = \lim_{a\in\mathcal{G}} \|P_{X_n}R|_{Y_n}\| = 0$$

and

$$\lim_{a\in\mathcal{G}} \|P_{\Sigma(a)X_n}\Gamma'(u_a)|_{\Sigma(a)Y_n}\| = \lim_{a\in\mathcal{G}} \|P_{\Sigma(a)X_n}(L-K-K_a)|_{\Sigma(a)Y_n}\| = \lim_{a\in\mathcal{G}} \|P_{\Sigma(a)X_n}R_a|_{\Sigma(a)Y_n}\| = 0$$

Moreover, from (3.20) and Lemma 3.6 we know that

$$\lim_{a\in\mathcal{G}} \|P_{X_{a,n}} - P_{X_n} - P_{\Sigma(a)X_n}\| = 0 \ .$$

Since $Y_n, \Sigma(a)Y_n \subseteq Y_{a,n}$, these identities imply

$$\lim_{a\in\mathcal{G}} \|P_{X_{a,n}}\Gamma'(u_a)|_{Y_{a_n}}\| \le \lim_{a\in\mathcal{G}} \|P_{X_n}\Gamma'(u_a)|_{Y_n}\| + \lim_{a\in\mathcal{G}} \|P_{\Sigma(a)X_n}\Gamma'(u_a)|_{\Sigma(a)Y_n}\| = 0$$

and prove (iv).

To show (v), note that by (iv) it suffices to prove

$$\liminf_{a\in\mathcal{G}} \inf_{y\in S_1Y_{a,n}} \|\Gamma'(u_a)y\| \ge \frac{1}{M} \ . \tag{3.57}$$

Thus suppose that (a_m) is a cofinal sequence in $\mathcal{G}$ and that (y_m) is a sequence in $S_1Y_{a,n}$. Extracting subsequences we may assume that

$$\begin{aligned} y_m &\rightharpoonup v \in Y_n \\ (-a_m) \star y_m &\rightharpoonup w \in Y_n \end{aligned} \tag{3.58}$$

since $(y_m) \subseteq Y_n$ and $((-a_m) \star y_m) \subseteq Y_n$. We set $z_m := y_m - v - a_m \star w$.

From here one proceeds exactly as in the proof of (3.16). Only note that now we have to use (3.58) to see $\|Rv\| \ge \|v\|/M$ and $\|Rw\| \ge \|w\|/M$, and that

$$\|Rz_m\| = \|RP_{Y_n}z_m\| + o(1) \ge \frac{1}{M}\|P_{Y_n}z_m\| + o(1) = \frac{1}{M}\|z_m\| + o(1)$$

since from $z_m \rightharpoonup 0$ it follows that $P_{X_n}z_m \to 0$ as $m \to \infty$. This proves (3.57) and thus (v). □

4. Mountain Pass Geometry

Recall the setting of Section 3. The application of Theorem 3.4 requires that we produce an isolated critical point with nonzero reduced local degree. In the present section we do this in the classical framework of mountain pass geometry [7] that arises if L is positive and Ψ in (3.1) is superquadratic. To keep the presentation short we do not strive for utmost generality here.

The assumptions in this section are (G3.1)–(G3.3), (F3.1)–(F3.5), (L3.1), (L3.2), and $\sigma(L) \subseteq \mathbb{R}^+$. By using a suitable equivalent scalar product $\langle \cdot, \cdot \rangle$ and an associated equivalent norm $\|\cdot\|$ on E we may assume that $L = I$ and

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \Psi(u) \ .$$

In addition we assume:

(G4.1) If $\mathcal{A} \subseteq \mathcal{G}$ contains no cofinal sequence then $\mathcal{A} \star u$ is relatively compact for every $u \in E$.

(F4.1) Ψ is weakly sequentially lower semicontinuous.

(F4.2) There is $\theta > 2$ such that $\Psi'(u)u \geq \theta\Psi(u) > 0$ for every $u \in E \setminus \{0\}$.

(F4.3) $\Psi''(u)[u, u] > \Psi'(u)u$ for all $u \in E \setminus \{0\}$.

(F4.4) If (u_n) is a bounded sequence in E and $a_n \star u_n \rightharpoonup 0$ as $n \to \infty$ for all sequences (a_n) in $\mathcal{G}$ then $\Psi'(u_n)u_n \to 0$.

Recall that we denote by $\mathcal{K}$ the set of nontrivial critical points of Φ. The following proposition yields the statement of Theorem 1.2 if it is combined with Theorem 3.4.

4.1 Proposition. *Under the hypotheses listed above, $\mathcal{K}$ is not empty, closed, and Φ achieves a positive minimum on $\mathcal{K}$. Moreover, denoting* $c_{\min} := \min \Phi(\mathcal{K}) > 0$*, every isolated critical point in $\mathcal{K}(c_{\min})$ has nonzero reduced local degree.*

Since these facts are more or less known the proof consists mainly of references to the literature. It will be given in Section 4.1, exactly keeping track of assumptions for better reference. This is necessary since the strongly indefinite case (handled in Section 5) relies on the results of the present section, under a different set of hypotheses.

Some remarks on the assumptions we impose on the action of $\mathcal{G}$ on E are in order. First, (G4.1) is clearly a consequence of (G3.4). On the other hand, consider the condition

(G4.2) The stabilizer of every u in $E \setminus \{0\}$ is finite.

Recall that the stabilizer of u in E is the set of a in $\mathcal{G}$ such that $a \star u = u$. Under our present assumptions (G3.4) follows from (G4.1) and (G4.2) if existence of an isolated critical point of Φ is assumed. To see this, suppose that $\bar{u}$ is an isolated critical point of Φ and that $\mathcal{A}$ is an infinite subset of $\mathcal{G}$. By invariance $\mathcal{G} \star \bar{u}$ has no accumulation point in E. If $\mathcal{A}$ contains no cofinal sequence, then by (G4.1) the set $\mathcal{A} \star \bar{u}$ is relative compact, and it is infinite by (G4.2), a contradiction. In our applications hypothesis (G4.2) is satisfied, so (G3.4) is necessary for the existence of *isolated* critical points. The main reason we do not assume (G3.4) in the present section is that we want to state Lemma 4.2 below under the weaker assumption (G4.1).

4.1. Proof of Proposition 4.1

First recall that from the BL-splitting property it follows that

(X4.1) $\Psi(0) = 0$, $\Psi'(0) = 0$, and $\Psi''(0) = 0$.

Using (F4.3) it is easy to verify:

(X4.2) If $u \in E \setminus \{0\}$ satisfies $\Phi'(u)u = 0$ then $\Phi''(u)[u,u] < 0$.

In [1, Lemma 4.2] it was shown that the following is a consequence of (F4.1) and (F4.2):

(X4.3) If Z is a finite-dimensional subspace of E then $\Phi(u) \to -\infty$ as $\|u\| \to \infty$, u in Z.

Next we establish the standard splitting lemma.

4.2 Lemma. *Recall that we have set $c_{\min} = \inf \Phi(\mathcal{K})$. It follows that $c_{\min} > 0$. For $c \in \mathbb{R}$ suppose that $(u_n) \subseteq E$ is a $(\mathrm{PS})_c$-sequence for Φ. Then either $c = 0$ and $u_n \to 0$ or $c \geq c_{\min}$ and there are $k \in \mathbb{N}$, $k \leq [c/c_{\min}]$, and for each $1 \leq i \leq k$ a sequence $(a_{i,n})_n \subseteq \mathcal{G}$ and an element $v_i \in \mathcal{K}$ such that, after extraction of a subsequence of (u_n),*

$$\left\| u_n - \sum_{i=1}^{k} a_{i,n} \star v_i \right\| \to 0$$

$$\Phi\left(\sum_{i=1}^{k} a_{i,n} \star v_i \right) \to \sum_{i=1}^{k} \Phi(v_i) = c$$

and $(a_{i,n} - a_{j,n})_n$ is cofinal for fixed $i \neq j$.

Proof. For a simple proof in an abstract setting see [1, Lemmata 4.3 and 4.5]. Only the last statement deserves explanation. If $u_n \rightharpoonup 0$ in E and if $(a_n) \subseteq \mathcal{G}$ contains no cofinal subsequence, then for every $v \in E$ the sequence $((-a_n) \star v)$ is relative compact by (G4.1). Hence $\langle a_n \star u_n, v\rangle = \langle u_n, (-a_n) \star v\rangle \to 0$. This shows that $a_n \star u_n \rightharpoonup 0$. With this fact in mind it is easy to transfer the proof to the present setting. $\square$

4.3 Remark. In what follows we will only make use of (X4.1)–(X4.3) and of Lemma 4.2.

We need to introduce some more notation and concepts. First denote

$$\dot{\Phi}^c := \{\, u \in E \mid \Phi(u) < c \,\}$$

for c in $\mathbb{R}$. Following Hofer [24] we say that a critical point $\bar{u}$ of Φ is of *mountain pass type* if for every small enough neighborhood U of $\bar{u}$ and $c = \Phi(\bar{u})$ the set $\dot{\Phi}^c \cap U$ is not empty and not path connected.

From (X4.1) and (X4.3) one concludes that Φ has Mountain Pass geometry in the following sense: $\Phi(0) = 0$, $\inf \Phi(S_r E) > 0$ for some $r > 0$, and there exists u in E with $\|u\| > r$ and $\Phi(u) \leq 0$. Hence there is a Palais-Smale sequence at a positive level (see e.g. [45, Theorem 1.15]) and Lemma 4.2 yields $\mathcal{K} \neq \varnothing$. Another application of Lemma 4.2 shows that Φ achieves its positive infimum $c_{\min}$ on $\mathcal{K}$.

For fixed u in E consider the map $g_u \colon \mathbb{R}_0^+ \to \mathbb{R}$ given by $g_u(t) := \Phi(tu)$. From (X4.1) and (X4.3) it follows that $g_u(0) = 0$, $g_u(t) > 0$ for small $t > 0$, and $g_u(t) \to -\infty$ as $t \to \infty$. Moreover, by (X4.2) $g_u''(t) < 0$ if $t > 0$ and $g_u'(t) = 0$. Hence there is a unique $t_u > 0$ such that $g_u'(t_u) = 0$, and g_u achieves its maximum in t_u. If $u \in \mathcal{K}$ then $t_u = 1$.

It follows from these facts that $\dot{\Phi}^{c_{\min}}$ has exactly two path connected components, one of them containing 0 (see e.g. the proof of [34, Lemma 3.1]). Moreover, every element in $\mathcal{K}(c_{\min})$ is of mountain pass type, and its Morse index is not zero.

Suppose now that $\bar{u}$ is an isolated critical point of Φ in $\mathcal{K}(c_{\min})$. The generalized Morse Lemma [14, Theorem 5.1] and the proof of [24, Theorem 2] yield that 0 is a strict local minimum of the reduction of Φ at $\bar{u}$ to $\mathcal{N}(\Gamma'(\bar{u}))$. Then $\operatorname{rdeg}_{\mathrm{loc}}(\Phi, \bar{u}) \neq 0$, as is well known (see e.g. [6, 37]). This finishes the proof. □

5. Strongly Indefinite Geometry

Keeping the notation of Section 3 we now turn to the case of indefinite L. The strategy is to assume convexity of Ψ, and to reduce the problem of finding an isolated critical point of Φ with nonvanishing reduced local degree to the mountain pass case handled in Section 4. This idea can be traced back to [5, 13] and was also used in [12].

Again we assume (G3.1)–(G3.3), (F3.1)–(F3.5) , (L3.1) and (L3.2). By a suitable change of scalar product and norm on E we may assume the following setting: We are given a splitting $E = E^+ \oplus E^-$ of E into orthogonal subspaces $E^\pm$ with associated bounded projections $P^\pm$. For $u \in E$ we write $u^\pm := P^\pm u$. The spaces $E^\pm$ are invariant under the action of $\mathcal{G}$, and the projections $P^\pm$ are equivariant. Moreover, $L = P^+ - P^-$ and

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Psi(u) .$$

From Section 4 we assume hypotheses (G4.1), (F4.2) and (F4.4). Moreover we make the assumptions that

(F5.1) Ψ is convex

(F5.2) $\Lambda' \colon E_w \to \mathcal{L}_s(E)$ is sequentially continuous at 0

(F5.3) There are $C \geq 0$ and a map $\kappa \colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with the following properties: $\lim_{t\to 0} \kappa(t) = 0$, $\kappa(t) \leq C(1 + t)$ for all $t \geq 0$, and for all $u \in E$ it holds that

$$\|\Psi'(u)\| \leq \kappa(\Psi'(u)u) .$$

(F5.4) For every $u \in E \setminus \{0\}$ and $v \in E$ it holds that

$$\Big(\Psi''(u)[u,u] - \Psi'(u)u\Big) + 2\Big(\Psi''(u)[u,v] - \Psi'(u)v\Big) + \Psi''(u)[v,v] > 0$$

The following theorem yields the statement of Theorem 1.2 if it is combined with Proposition 4.1, Remark 4.3 and Theorem 3.4. Note that for the restricted group action of $\mathcal{G}$ on E^+ (G3.1)–(G3.3) and (G4.1) are also satisfied (replacing E by E^+).

5.1 Theorem. *There is a map h in $C^1(E^+, E^-)$ that is uniquely defined by either one of the following properties: For $u \in E^+$ and $v \in E^-$*

$$v \neq h(u) \quad \Leftrightarrow \quad \Phi(u+v) < \Phi(u+h(u)) \tag{5.1}$$

$$v = h(u) \quad \Leftrightarrow \quad P^-\Gamma(u+v) = 0\ . \tag{5.2}$$

Define $\Phi_r\colon E^+ \to \mathbb{R}$ by $\Phi_r(u) := \Phi(u+h(u))$ and let Γ_r denote the gradient of Φ_r. Then we have:

a) *$h \in C^{1+\alpha}(E^+, E^-)$ uniformly on bounded subsets.*

b) *Critical points of Φ_r and Φ are in one to one correspondence via the injective map $u \mapsto u + h(u)$ from E^+ into E.*

c) *Φ_r has the form $\Phi_r(u) = \frac{1}{2}\|u\|^2 - \Psi_r(u)$ where, replacing E by E^+, Φ by Φ_r and Ψ by Ψ_r, conditions* (F3.1)–(F3.5) *and* (X4.1)–(X4.3) *apply. In addition, Lemma* 4.2 *is valid.*

The proof will be given in Section 5.2.

5.1. More on the BL-splitting Property

Here we collect some results that allow us to prove the BL-splitting property for compositions of BL-splitting maps.

5.2 Definition. Suppose that X, Y and Z are Banach spaces and that $K\colon X \to \mathcal{L}(Y,Z)$ is a map. We say that K *satisfies condition* (K) if the following hold

(i) K BL-splits

(ii) K is bounded

(iii) $K(x)$ is a compact operator for all $x \in X$

(iv) $K\colon X_w \to \mathcal{L}_s(Y,Z)$ and $K^*\colon X_w \to \mathcal{L}_s(Z^*,Y^*)$ are sequentially continuous at 0.

5.3 Lemma. *Suppose that W, X, Y and Z are Banach spaces, $K_1\colon W \to \mathcal{L}(X,Y)$, $K_2\colon W \to \mathcal{L}(Y,Z)$, and K_1 and K_2 satisfy* (K). *Then $K\colon W \to \mathcal{L}(X,Z)$ defined by $K(w) := K_2(w)K_1(w)$ satisfies* (K).

Proof. First we show that K BL-splits. Suppose that $w_n \rightharpoonup w$ in W. Take a sequence (x_n) in S_1X with

$$\|K_2(w)K_1(w_n - w)x_n\| \geq \|K_2(w)K_1(w_n - w)\| - \frac{1}{n} \tag{5.3}$$

for all n. For every $y^* \in Y^*$ we obtain from (iv) of condition (K) for K_1^* and from $K_1(0) = 0$ that $K_1^*(w_n - w)y^* \to 0$ and thus

$$y^*[K_1(w_n - w)x_n] = K_1^*(w_n - w)[y^*][x_n] \to 0$$

since (x_n) is bounded. Hence $K_1(w_n - w)x_n \rightharpoonup 0$ in Y, so $K_2(w)K_1(w_n - w)x_n \to 0$ by the compactness of $K_2(w)$. Together with (5.3) it follows that

$$\|K_2(w)K_1(w_n - w)\|_{\mathcal{L}(X,Z)} \to 0 \quad \text{as } n \to \infty. \tag{5.4}$$

Take a sequence (x_n) in S_1X with

$$\|K_2(w_n - w)K_1(w)x_n\| \geq \|K_2(w_n - w)K_1(w)\| - \frac{1}{n} \tag{5.5}$$

for all n. By compactness of $K_1(w)$, passing to a subsequence we may assume that $K_1(w)x_n$ converges in Y. Now the boundedness of K_2, (iv) of condition (K) for K_2, and $K_2(0) = 0$ imply that $K_2(w_n - w)K_1(w)x_n \to 0$. Hence (5.5) yields

$$\|K_2(w_n - w)K_1(w)\|_{\mathcal{L}(X,Z)} \to 0 \quad \text{as } n \to \infty. \tag{5.6}$$

Using (5.4), (5.6), and (i) and (ii) of (K) for K_1 and K_2 we obtain

$$\begin{aligned} K(w_n) = (K_2(w) + K_2(w_n - w))(K_1(w) + K_1(w_n - w)) + o(1) \\ = K(w) + K(w_n - w) + o(1) \end{aligned}$$

as $n \to \infty$ and hence the BL-splitting property for K. Routine checks show that K also satisfies (ii), (iii) and (iv) of condition (K). ☐

The next lemma is a straightforward consequence of the spectral theorem.

5.4 Lemma. *Suppose that Z is a Hilbert space, $K \in \mathcal{L}(Z)$ is compact, selfadjoint, and $\sigma(K) \subseteq [0, \infty)$. Then $(I + K)$ is invertible and $\|(I + K)^{-1}\| \leq 1$. Setting $L := I - (I + K)^{-1}$ we have $\|L\| \leq 1$ and $\|Lz\| \leq 2\|Kz\|$ for every $z \in Z$.*

5.5 Lemma. *Suppose that X is a Banach space, Z a Hilbert space, $K\colon X \to \mathcal{L}(Z)$ satisfies* (K)*, and $K(x)$ is selfadjoint with $\sigma(K(x)) \subseteq [0, \infty)$ for every $x \in X$. Define $L\colon X \to \mathcal{L}(Z)$ by $L(x) := I - (I + K(x))^{-1}$. Then L satisfies condition* (K).

Proof. From Lemma 5.4 and from the selfadjointness of K and L (ii)–(iv) of condition (K) for L follow at once. Therefore it only remains to show the BL-splitting property for L. Suppose that $x_n \rightharpoonup x$ in X. From condition (K) for K it follows as in the proof of (5.4) and (5.6) that

$$\begin{aligned} &\|K(x)K(x_n - x)\| \to 0 \\ &\|K(x_n - x)K(x)\| \to 0 \end{aligned} \tag{5.7}$$

as $n \to \infty$. Moreover by BL-splitting and boundedness of K

$$\begin{aligned} &K(x_n)K(x_n - x) - K(x_n - x)K(x_n) \\ &\qquad = K(x)K(x_n - x) - K(x_n - x)K(x) + o(1) = o(1) \; . \end{aligned}$$

Hence

$$K(x_n)K(x_n - x) = K(x_n - x)K(x_n) + o(1)$$

and similarly

$$K(x_n)K(x) = K(x)K(x_n) + o(1) \; . \tag{5.8}$$

Now set $a_n := K(x_n)$, $b := K(x)$ and $c_n := K(x_n - x)$. These linear operators are uniformly bounded since K is a bounded map. In the following straightforward computation we will thus freely commute a_n, b and c_n a finite number of times, only adding terms $o(1)$ by (5.7)–(5.8):

$$\begin{aligned} &(I + a_n)(I + b)(I + c_n)(L(x_n) - L(x) - L(x_n - x)) \\ &\qquad = (I + a_n)(I + b)(I + c_n)(-I + (I + b)^{-1} + (I + c_n)^{-1} - (I + a_n)^{-1}) \\ &\qquad = a_n - b - c_n - (2I + a_n)bc_n + o(1) \\ &\qquad = o(1) \; . \end{aligned}$$

Here the last equality holds since K BL-splits and by (5.7). Note that by Lemma 5.4 $\|(I+a_n)^{-1}\|$, $\|(I+b)^{-1}\|$ and $\|(I+c_n)^{-1}\|$ remain bounded by 1, so we can conclude. $\square$

5.2. Proof of Theorem 5.1

We start by constructing the map h. For fixed $u \in E^+$ define $\varphi_u \colon E^- \to \mathbb{R}$ by

$$\varphi_u(v) := \Phi(u + v) = \frac{1}{2}(\|u\|^2 - \|v\|^2) - \Psi(u + v) \; .$$

From the convexity of Ψ it follows that

$$\Psi''(u)[v, v] \geq 0 \tag{5.9}$$

for all u, v in E, and hence

$$\varphi_u''(v)[w, w] = \Phi''(u + v)[w, w] = -\|w\|^2 - \Psi''(u + v)[w, w] \leq -\|w\|^2 \tag{5.10}$$

for all v, w in E^-. Moreover

$$\varphi_u(v) \le \frac{1}{2}(\|u\|^2 - \|v\|^2)$$

since $\Psi \ge 0$. Therefore φ_u is strictly concave and $\lim_{\|v\|\to\infty} \varphi_u(v) = -\infty$. From weak sequential upper semicontinuity of φ_u it follows that there is a unique strict maximum point $h(u)$ for φ_u, which is also the only critical point of φ_u on E^-. This proves (5.1) and (5.2).

For later use we note that (5.10) and $\varphi_u'(h(u)) = 0$ imply for all $u \in E^+$ and $v \in E^-$ that

$$\begin{aligned}\varphi_u(v) - \varphi_u(h(u)) &= \int_0^1 (1-t)\varphi_u''(h(u) + t(v - h(u)))[v - h(u), v - h(u)]\, \mathrm{d}t \\ &\le -\frac{1}{2}\|v - h(u)\|^2\end{aligned}$$

and hence

$$\|h(u) - v\|^2 \le 2(\Phi(u + h(u)) - \Phi(u + v)) \; . \tag{5.11}$$

From (5.10) it follows that $P^-\Gamma'(u + h(u))|_{E^-}$ is an isomorphism with

$$\|(P^-\Gamma'(u + h(u))|_{E^-})^{-1}\| \le 1 \tag{5.12}$$

for every $u \in E^+$. Hence Lemma 2.4 yields that locally $h \in C^{1+\alpha}$ and

$$\begin{aligned} h'(u) &= -(P^-\Gamma'(u + h(u))|_{E^-})^{-1} P^-\Gamma'(u + h(u))|_{E^+} \\ &= -(I_{E^-} + P^-K(u)|_{E^-})^{-1} P^-K(u)|_{E^+} \\ &= (I_{E^-} - (I_{E^-} + P^-K(u)|_{E^-})^{-1}) P^-K(u)|_{E^+} - P^-K(u)|_{E^+} \end{aligned} \tag{5.13}$$

Here we have set

$$K(u) := \Lambda'(u + h(u)) \tag{5.14}$$

for $u \in E^+$. Moreover we see from Lemma 2.4 and $\Phi'(0) = 0$ that

$$h(0) = 0$$

and hence by (5.13) that

$$h'(0) = 0 \; .$$

Observe that by (F3.1), (F3.2), (F3.4), (F5.2), and by the selfadjointness of $\Lambda'(u)$ for every $u \in E$ the map Λ' satisfies condition (K).

The next Lemma implies a) of Theorem 5.1.

5.6 Lemma. (i) *h is equivariant under $\mathcal{G}$.*

(ii) *The map h is in $C^{1+\alpha}(E^+, E^-)$, uniformly on bounded subsets.*

(iii) *h is weakly sequentially continuous and BL-splits.*

(iv) *h' satisfies condition* (K).

Proof. (i) If $u \in E^+$ and $a \in \mathcal{G}$ we have by invariance of Φ and by (5.1)

$$\begin{aligned}\Phi(a \star u + h(a \star u)) = \Phi(u + (-a) \star h(a \star u)) \leq \Phi(u + h(u)) \\ = \Phi(a \star u + a \star h(u)) \leq \Phi(a \star u + h(a \star u)) \ .\end{aligned}$$

Hence the inequalities are in fact equalities, and

$$\Phi(a \star u + a \star h(u)) = \Phi(a \star u + h(a \star u))$$

together with (5.1) implies that $a \star h(u) = h(a \star u)$.

(ii) For $u \in E^+$ we obtain from (5.1) and $\Psi \geq 0$ that

$$0 \leq \Phi(u + h(u)) - \Phi(u) = -\frac{1}{2}\|h(u)\|^2 + \Psi(u) - \Psi(u + h(u)) \leq -\frac{1}{2}\|h(u)\|^2 + \Psi(u) \ .$$

Hence the boundedness of Ψ implies that of h. Now the boundedness of h and Λ' imply the boundedness of h' in view of (5.12) and (5.13). Moreover boundedness of h, (5.12), (F3.1), and Lemma 2.4 imply that for each $r_1 \geq 0$ there are $C \geq 0$ and $r_2 \geq 0$ such that

$$\|h'(u) - h'(v)\| \leq C\|u - v\|^\alpha$$

holds whenever $u, v \in E^+$, $\|u - v\| \leq r_2$, and $\|u\|, \|v\| \leq r_1$. Together with the boundedness of h' this yields uniform Hölder continuity of h' with exponent α on bounded subsets of E^+.

(iii) First we claim that

$$(5.15) \qquad h \text{ is weakly sequentially continuous at } 0.$$

To see this suppose that $u_n \rightharpoonup 0$ in E^+. Since h is bounded, passing to a subsequence we may assume that $h(u_n) \rightharpoonup v$ in E^-. Then $u_n + h(u_n) \rightharpoonup v$. Now (5.1) together with $\Psi \geq 0$ and the BL-splitting property of Φ implies that

$$\frac{1}{2}\|v\|^2 \leq -\Phi(v) = \Phi(u_n + h(u_n) - v) - \Phi(u_n + h(u_n)) + o(1) \leq o(1)$$

as $n \to \infty$. Hence $v = 0$, and (5.15) is proved since $h(0) = 0$.

Next we show that

$$(5.16) \qquad h \text{ BL-splits.}$$

Suppose therefore that $u_n \rightharpoonup u$ in E^+. We may again assume that $h(u_n) \rightharpoonup v$ in E^-. Note that $h(u_n - u) \rightharpoonup h(0) = 0$ by (5.15). Using that Φ BL-splits, we therefore obtain

$$\begin{aligned}\Phi(u_n + h(u_n)) &= \Phi(u + v) + \Phi(u_n - u + h(u_n) - v) + o(1) \\ &\leq \Phi(u + h(u)) + \Phi(u_n - u + h(u_n - u)) + o(1) && \text{by (5.1)} \\ &= \Phi(u_n + h(u) + h(u_n - u)) + o(1) && \text{by (5.15)}\end{aligned}$$

as $n \to \infty$. Together with (5.11) it now follows that

$$\|h(u_n) - h(u) - h(u_n - u)\|^2 \le 2(\Phi(u_n + h(u_n)) - \Phi(u_n + h(u) + h(u_n - u))) \le o(1)$$

and (5.16) is proved.

It is clear that (5.15) and (5.16) imply that

$$h \text{ is weakly sequentially continuous.}$$

(iv) Since Λ' satisfies (K), h BL-splits, and h is bounded and weakly sequentially continuous, it is straightforward to see that K as defined in (5.14) also satisfies condition (K). Hence the claim follows from (5.9), (5.13), and Lemmata 5.3 and 5.5. □

Define Φ_r and Γ_r as in the statement of the theorem. From (5.2) it is clear that b) of Theorem 5.1 holds. Moreover it is easy to see that $\Phi_r \in C^{2+\alpha}(E^+, \mathbb{R})$ and

$$\Gamma_r(u) = \Gamma(u + h(u)) = P^+\Gamma(u + h(u)) \ . \tag{5.17}$$

$$\Gamma_r'(u) = \Gamma'(u + h(u))(I + h'(u)) = P^+\Gamma'(u + h(u))(I + h'(u)) \tag{5.18}$$

We now turn to the proof of c). Set

$$\Psi_r(u) := \frac{1}{2}\|h(u)\|^2 + \Psi(u + h(u))$$

for u in E^+. It follows that $\Phi_r(u) = \frac{1}{2}\|u\|^2 - \Psi_r(u)$. From (5.2) we obtain for u in E^+ that

$$P^-\Lambda(u + h(u)) = -h(u) \tag{5.19}$$

and hence for all v in E^+

$$\Psi_r'(u)v = \langle h(u), h'(u)v\rangle + \langle \Lambda(u + h(u)), v + h'(u)v\rangle = \langle P^+\Lambda(u + h(u)), v\rangle \ .$$

Denoting by Λ_r the gradient of Ψ_r this yields

$$\Lambda_r(u) = P^+\Lambda(u + h(u)) \tag{5.20}$$

$$\Lambda_r'(u) = P^+K(u)|_{E^+} + P^+K(u)h'(u) \ . \tag{5.21}$$

Using the properties of Ψ, K and h it is straightforward to check that (F3.1)–(F3.5) and (X4.1) hold if E is replaced by E^+, Φ is replaced by Φ_r and Ψ is replaced by Ψ_r.

To see (X4.2) fix x in $E^+ \setminus \{0\}$ with $\Phi_r'(x)x = 0$, and set $u = x + h(x)$ and $v = h'(x)x - h(x) \in E^-$. Then $u \ne 0$ and by (5.19)

$$\begin{aligned} 0 = \Phi_r'(x)x &= \|x\|^2 - \Psi_r'(x)x = \|x\|^2 - \langle \Lambda(u), x\rangle \\ &= \|x\|^2 - \|h(x)\|^2 - \langle \Lambda(u), x + h(x)\rangle = \|u^+\|^2 - \|u^-\|^2 - \langle \Lambda(u), u\rangle \end{aligned} \tag{5.22}$$

and

$$P^-\Lambda(u) = -u^- \ . \tag{5.23}$$

We now calculate using $u \neq 0$, $v \in E^-$, (5.18), (5.19), (5.22), (5.23) and (F5.4):

$$
\begin{aligned}
\Phi_r''(x)[x,x] &= \langle \Gamma_r'(x)x, x\rangle \\
&= \langle \Gamma'(u)[x + h'(x)x], x + h'(x)x\rangle \\
&= \langle \Gamma'(u)[u+v], u+v\rangle \\
&= \langle \Gamma'(u)u, u\rangle + 2\langle \Gamma'(u)u, v\rangle + \langle \Gamma'(u)v, v\rangle \\
&= \|u^+\|^2 - \|u^-\|^2 - \langle \Lambda'(u)u, u\rangle + 2\langle u^+ - u^- - \Lambda'(u)u, v\rangle \\
&\qquad\qquad\qquad\qquad + \langle -v - \Lambda'(u)v, v\rangle \\
&= \langle \Lambda(u) - \Lambda'(u)u, u\rangle + 2\langle \Lambda(u) - \Lambda'(u)u, v\rangle - \langle \Lambda'(u)v, v\rangle - \|v\|^2 \\
&= (\Psi'(u)u - \Psi''(u)[u,u]) + 2(\Psi'(u)v - \Psi''(u)[u,v]) \\
&\qquad\qquad\qquad\qquad - \Psi''(u)[v,v] - \|v\|^2 \\
&< 0\,.
\end{aligned}
$$

This proves (X4.2).

5.7 Remark. The above computation using condition (F5.4) goes back to an idea of Pankov [35]. It was also used in [29].

Turning to the proof of (X4.3) for Φ_r defined on E^+ suppose that Z is a finite-dimensional subspace of E^+. If $u \in Z$ then $u + h(u) \in Z \oplus E^-$, and $\|u + h(u)\| \to \infty$ as $\|u\| \to \infty$. Observe that the convexity of Ψ implies its weak sequential lower semicontinuity. This fact together with (F4.2) is sufficient to use Lemma 4.2 in [1]. Applying this lemma we obtain that

$$\Phi_r(u) = \Phi(u + h(u)) \to -\infty$$

as $\|u\| \to \infty$ and $u \in Z$.

It only remains to prove the assertion of Lemma 4.2 for Φ_r. Set

$$\mathcal{K}_r := \{\, u \in E^+ \smallsetminus \{0\} \mid \Phi_r'(u) = 0 \,\}\,.$$

By b) of Theorem 5.1 $\mathcal{K} = \{\, u + h(u) \mid u \in \mathcal{K}_r \,\}$ and $\mathcal{K}_r = P^+\mathcal{K}$. It was shown in [1, Lemma 4.3] that $\inf \Phi(\mathcal{K}) > 0$. From $\inf \Phi_r(\mathcal{K}_r) = \inf \Phi(\mathcal{K})$ it follows that $c_{\min} := \inf \Phi_r(\mathcal{K}_r) > 0$. Suppose now that $c \in \mathbb{R}$ and that $(x_n) \subseteq E^+$ is a $(PS)_c$-sequence for Φ_r. Since $P^-\Gamma(x_n + h(x_n)) = 0$ it follows immediately from (5.17) that $u_n := x_n + h(x_n)$ defines a $(PS)_c$-sequence for Φ. We can apply Lemma 4.2 for Φ, which can be proved under our present conditions on Φ (see [1]). Hence either $c = 0$ or $c \geq c_{\min}$. In the first case $u_n \to 0$ and $x_n = P^+u_n \to 0$ as $n \to \infty$. In the second case let k in $\mathbb{N}$, $(a_{i,n})_n$ in $\mathcal{G}$ and v_i in $\mathcal{K}$ be given with the properties stated in Lemma 4.2. Set $y_i := P^+v_i$, so $v_i = y_i + h(y_i)$ and $y_i \in \mathcal{K}_r$. Clearly

$$\Big\| x_n - \sum_{i=1}^k a_{i,n} \star y_i \Big\| = \Big\| P^+\Big(u_n - \sum_{i=1}^k a_{i,n} \star v_i\Big) \Big\| = o(1)$$

as $n \to \infty$. Moreover

$$c = \sum_{i=1}^k \Phi(v_i) = \sum_{i=1}^k \Phi_r(y_i) = \Phi_r\Big(\sum_{i=1}^k a_{i,n} \star y_i\Big) + o(1)$$

as $n \to \infty$ since Φ_r BL-splits and is $\mathcal{G}$-invariant. This finishes the proof of Theorem 5.1.

6. Applications

To apply the abstract theorems proved in the preceding sections we now analyze the relevant properties of the variational functionals involved. As in the introduction let us denote $E := H^1(\mathbb{R}^N)$, and let T denote the unique selfadjoint operator induced on $L^2(\mathbb{R}^N)$ by $-\Delta + V$. Moreover, assume condition (A1.1). In what follows, for $t > 0$ we write $L^t := L^t(\mathbb{R}^N)$.

6.1. The Group Action

Recall the definition of the action of $\mathbb{Z}^N$ on E by translation, as described in Section 1.3. We define $\mathcal{G} := \mathbb{Z}^N$ and define the direction $\succ$ on $\mathcal{G}$ as follows: If $a, b \in \mathcal{G}$, then $a \succ b$ if and only if $|a| \geq |b|$. It is clear that then (G3.1)–(G3.4), (G4.1) and (G4.2) hold for the action of $\mathcal{G}$ on E.

6.2. The Quadratic Part

Denote $E^\pm := E \cap (L^2)^\pm$, where $(L^2)^\pm$ are the generalized eigenspaces of T in L^2 corresponding to the positive and negative part of $\sigma(T)$. Of course, if $\sigma(T) \subseteq \mathbb{R}^+$ then $E^- = \{0\}$. Denote by $P^\pm$ the pair of bounded projections induced by the splitting $E = E^+ \oplus E^-$. For u in E we write $u^\pm := P^\pm u$. The projections $P^\pm$ are equivariant and the spaces $E^\pm$ invariant under the action of $\mathbb{Z}^N$.

As is often done we endow E with the scalar product

$$\langle u, v \rangle := (|T|^{1/2}u, |T|^{1/2}v)$$

The projections $P^\pm$ are orthogonal with respect to $\langle \cdot\ , \cdot \rangle$, and the norm induced by this new scalar product will be denoted by $\|\cdot\|$. It is equivalent to the original norm on $H^1(\mathbb{R}^N)$ introduced in Section 1.5. We can now write

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x = \|u^+\|^2 - \|u^-\|^2\ .$$

6.3. Analysis of Multiplication and Superposition Operators

The proof of regularity, compactness and BL-splitting properties of the superquadratic part in the energy functional will be based on the following technical lemmata.

6.1 Lemma. *Suppose that $s, t, \mu \geq 1$ are given with*

$$\frac{1}{s} + \frac{1}{t} = \frac{1}{\mu}\ .$$

Then the bilinear map

$$\begin{aligned} L^s \times L^t &\to L^\mu \\ (u, v) &\mapsto uv \end{aligned}$$

is bounded with $|uv|_\mu \le |u|_s|v|_t$. *If* (u_n) *and* (v_n) *are bounded sequences in* L^s *and* L^t *respectively, if* $u \in L^s$ *and* $v \in L^t$, $u_n \to u$ *in* L^s *and* $v_n \to v$ *in* L^t_{loc}, *then* $u_n v_n \to uv$ *in* L^μ.

We omit the easy proof. For the second statement see also the proof of [1, Lemma 3.1].

6.2 Lemma. *Suppose we are given* $r, s, t \in [1, \infty)$, $U \in L^r$, *such that*

$$\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{t}\ .$$

Then the linear operator

$$\begin{aligned} L^s &\to L^t \\ u &\mapsto U * u \end{aligned}$$

is bounded and $|U * u|_t \le |U|_r |u|_s$. *If* u *is in* L^s *and if* (u_n) *is a bounded sequence in* L^s *such that* $u_n \to u$ *in* L^s_{loc}, *then* $(U * u_n)$ *is bounded in* L^t *and* $U * u_n \to U * u$ *in* L^t_{loc} *as* $n \to \infty$.

The preceding lemma was also proved in [1, Lemma 3.1].

Now we formulate yet another variant of the well-known Brezis-Lieb Lemma [11]. A similar statement was proved in [1, Lemma 3.2].

6.3 Lemma. *Suppose we are given* $t \ge 1$, $\mu > 0$ *such that* $t\mu \ge 1$. *Suppose that* $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *is a Caratheodory function such that there is* $C \ge 0$ *with*

$$|f(x, u)| \le C|u|^\mu$$

for all x *and* u. *Denote by* Σ_f *the (continuous) superposition operator induced by* f, *mapping* $L^{t\mu}$ *into* L^t, *and assume that* Σ_f *is uniformly continuous on bounded subsets. Then for every bounded sequence* (u_n) *in* $L^{t\mu}$ *that converges in* $L^{t\mu}_{\mathrm{loc}}$ *to some* $u \in L^{t\mu}$ *it holds that*

$$\Sigma_f(u_n) - \Sigma_f(u_n - u) \to \Sigma_f(u) \qquad \text{in } L^t$$

as $n \to \infty$.

Proof. Assume by contradiction that after passing to a subsequence it holds that

$$\liminf_{n\to\infty} \|\Sigma_f(u_n) - \Sigma_f(u_n - u) - \Sigma_f(u)\|_{L^t} > 0\ . \tag{6.1}$$

Define functions $Q_n\colon [0, \infty) \to [0, \infty)$ by

$$Q_n(R) := \int_{B_R} |u_n|^{t\mu}\, \mathrm{d}x\ .$$

Then the functions Q_n are uniformly bounded and nondecreasing. Passing to a subsequence we may assume that (Q_n) converges pointwise almost everywhere to a bounded nondecreasing function Q [30]. Again passing to a subsequence it is easy to build a sequence $R_n \to \infty$ such that for every $\varepsilon > 0$ there is $R > 0$, arbitrarily large, with

$$\limsup_{n\to\infty}(Q_n(R_n) - Q_n(R)) \le \varepsilon \, .$$

Hence for fixed $\varepsilon > 0$ we may choose $R > 0$ such that

$$\limsup_{n\to\infty}\int_{B_{R_n}\setminus B_R}|u_n|^{t\mu}\,\mathrm{d}x \le \varepsilon \qquad \text{and} \qquad \int_{\mathbb{R}^N\setminus B_R}|u|^{t\mu}\,\mathrm{d}x \le \varepsilon \, .$$

Set $v_n := \chi_{B_{R_n}} u$. From the continuity of Σ_f on $L^{t\mu}(B_R)$ we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_{B_R}|f(x,u_n) - f(x,u_n - v_n) - f(x,v_n)|^t\,\mathrm{d}x \\
&\qquad\qquad = \lim_{n\to\infty}\int_{B_R}|f(x,u_n) - f(x,u_n - u) - f(x,u)|^t\,\mathrm{d}x = 0 \, .
\end{aligned}$$

From this it follows that

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{\mathbb{R}^N}|f(x,u_n) - f(x,u_n - v_n) - f(x,v_n)|^t\,\mathrm{d}x \\
&\qquad = \limsup_{n\to\infty}\int_{B_{R_n}\setminus B_R}|f(x,u_n) - f(x,u_n - u) - f(x,u)|^t\,\mathrm{d}x \\
&\qquad \le C\limsup_{n\to\infty}\int_{B_{R_n}\setminus B_R}(|u_n|^\mu + |u_n - u|^\mu + |u|^\mu)^t\,\mathrm{d}x \\
&\qquad \le C\limsup_{n\to\infty}\int_{B_{R_n}\setminus B_R}(|u_n|^{t\mu} + |u|^{t\mu})\,\mathrm{d}x \\
&\qquad \le C\varepsilon
\end{aligned}$$

where C is independent of ε. Letting ε tend to 0 and using that $v_n \to u$ in $L^{t\mu}$ we obtain

$$\Sigma_f(u_n) - \Sigma_f(u_n - v_n) - \Sigma_f(u) \to 0 \qquad \text{in } L^t \, .$$

Hence by (6.1)

$$\liminf_{n\to\infty}\|\Sigma_f(u_n - v_n) - \Sigma_f(u_n - u)\|_{L^t} = \liminf_{n\to\infty}\|\Sigma_f(u_n) - \Sigma_f(u) - \Sigma_f(u_n - u)\|_{L^t} > 0 \, ,$$

in contradiction with the uniform continuity of Σ_f on bounded subsets of $L^{t\mu}$. □

6.4. The Local Equation

Recall the assumptions (A1.2)–(A1.4) we have required on f. Also recall the embeddings $E \to L^p$ for $p \in [2, 2^*)$. It holds that if $u_n \rightharpoonup u$ in E then (u_n) is bounded in L^p and converges to u in L^p_{loc}, for $p \in [2, 2^*)$. The nonlinearity can be written as $f = f^1 + f^2$ where

$$|f^i_{uu}(x,u)| \le C(|u|^{p_i - 3}) \tag{6.2}$$

for $i = 1, 2$. Therefore it is easy to prove (F3.1)–(F3.5), (F4.1)–(F4.3), (F5.1) and (F5.2) for $\Psi\colon E \to \mathbb{R}$ defined by

$$\Psi(u) := \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x\ ,$$

using Lemmata 6.1 and 6.3. One should keep in mind here that the composition of a BL-splitting map with a bounded linear operator also BL-splits.

To see that (F4.4) holds, suppose that (u_n) is a bounded sequence in E such that $a_n \star u_n \rightharpoonup 0$ as $n \to \infty$ for every sequence (a_n) in $\mathcal{G}$. It follows that $a_n \star u_n \to 0$ in L^p_{loc} for every sequence (a_n) in $\mathcal{G}$ and every $p \in [2, 2^*)$. Hence

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^N} \int_{B_R(x)} |u_n|^p\,\mathrm{d}x = 0$$

for every $R > 0$ and $p \in [2, 2^*)$. Now the Vanishing Lemma of Lions [31, Lemma I.1] implies that $u_n \to 0$ as $n \to \infty$ in L^p for every $p \in (2, 2^*)$, and so $\Psi'(u_n)[u_n] \to 0$.

For the splitting $f = f^1 + f^2$ introduced above we easily obtain from (6.2) that

$$|f^i(x,u)|^{p_i'} \le C f^i(x,u)u$$

for $i = 1, 2$. Here p_i' denotes the Hölder exponent conjugate to p_i. Therefore

$$\|\Psi'(u)\|_{E^*} \le C\left(\sqrt{\Psi'(u)u} + \Psi'(u)u\right)$$

and (F5.3) is satisfied.

It remains to prove (F5.4). First note that for $a, b, c \in \mathbb{R}$ with $a, c \ge 0$ we have the implications

$$b^2 \le ac \quad \Rightarrow \quad a + 2b + c \ge 0 \tag{6.3}$$

$$b^2 < ac \quad \Rightarrow \quad a + 2b + c > 0\ . \tag{6.4}$$

Consider some fixed x in $\mathbb{R}^N$ and $u, v \in \mathbb{R}$. For convenience set $f := f(x,u)$, $f' := f_u(x,u)$ and $g := (f'u^2 - fu) + 2(f'u - f)v + f'v^2$. Then $u = 0$ implies $g = 0$, and $u \ne 0$ and $v = 0$ implies $g > 0$ by (A1.4). If $u, v \ne 0$ we find from (A1.4) that $f/u > 0$. Together with (A1.4) again this implies $(f'u - f)^2v^2 < (f'u^2 - fu)f'v^2$, and hence $g > 0$ by (6.4). All in all we see that $g \ge 0$, $g > 0$ if $u \ne 0$, and therefore, if $u \in E \setminus \{0\}$ and $v \in E$, we have

$$\Big(\Psi''(u)[u,u] - \Psi'(u)u\Big) + 2\Big(\Psi''(u)[u,v] - \Psi'(u)v\Big) + \Psi''(u)[v,v]$$
$$= \int_{\mathbb{R}^N} \Big((f'u^2 - fu) + 2(f'u - f)v + f'v^2\Big) > 0\ .$$

Having proved all the necessary assumptions on Φ, Theorem 1.2 for (L) is a consequence of Theorem 3.4, Proposition 4.1 and Theorem 5.1.

6.5. The Nonlocal Equation

In the setting of the nonlocal equation we assume conditions (A1.5)–(A1.7) and define $\Psi\colon E \to \mathbb{R}$ by

$$\Psi(u) := \frac{1}{4}\int_{\mathbb{R}^3} (W * u^2)u^2 \,\mathrm{d}x \,.$$

To facilitate the calculations, we introduce the bilinear expression

$$I(u,v) := \int_{\mathbb{R}^N} (W * u)v \,\mathrm{d}x$$

for appropriate measurable functions u, v on $\mathbb{R}^N$. It is symmetric since W is even. Then for $u, v \in E$:

$$\Psi(u) = \frac{1}{4} I(u^2, u^2) \tag{6.5}$$

$$\Psi'(u)v = I(u^2, uv) \tag{6.6}$$

$$\Psi''(u)[v,w] = 2I(uv, uw) + I(u^2, vw) \tag{6.7}$$

$$I(u^2, u^2) > 0 \qquad \text{if } u \neq 0, \text{ by (A1.6)} \tag{6.8}$$

$$I(u,v) \geq 0 \qquad \text{if } u, v \geq 0, \text{ since } W \geq 0 \tag{6.9}$$

Combining Lemmata 6.1, 6.2 and 6.3 it is not difficult to prove properties (F3.1)–(F3.5), (F4.1)–(F4.4), and (F5.2) similarly as in Section 6.4. To prove the BL-splitting property for Ψ'' one can apply Lemma 5.3.

Conditions (F5.1)–(F5.4) need only be shown if $\sigma(T) \cap \mathbb{R}^- \neq \varnothing$, since otherwise the results from Section 4 apply. Therefore assume that W is positive definite (see (A1.7)). For appropriate measurable functions u, v on $\mathbb{R}^N$ it holds that

$$I(u,u) \geq 0 \qquad \text{since } W \text{ is positive definite} \tag{6.10}$$

$$|I(u,v)| \leq \sqrt{I(u,u)}\sqrt{I(v,v)} \qquad \text{by (6.10)} \,. \tag{6.11}$$

For all $u, v \in E$ it follows that

$$\Psi''(u)[v,v] = 2I(uv, uv) + I(u^2, v^2) \geq 0$$

from (6.9) and (6.10). Hence Ψ is convex and (F5.1) is satisfied. The proof of (F5.3) can be found in [1, Lemma 3.6].

To see that (F5.4) holds, consider u in $E \setminus \{0\}$ and v in E. Then

$$\begin{aligned}
&\Big(\Psi''(u)[u,u] - \Psi'(u)u\Big) + 2\Big(\Psi''(u)[u,v] - \Psi'(u)v\Big) + \Psi''(u)[v,v] \\
&\qquad = 2I(u^2,u^2) + 4I(u^2,uv) + 2I(uv,uv) + I(u^2,v^2) \\
&\qquad > I(u^2,(u+v)^2) + \frac{1}{2}I(u^2,u^2) + 2I(u^2,uv) + 2I(uv,uv) && \text{by (6.8)} \\
&\qquad \geq \frac{1}{2}I(u^2,u^2) + 2I(u^2,uv) + 2I(uv,uv) && \text{by (6.9)} \\
&\qquad \geq \frac{1}{2}I(u^2,u^2) - 2\sqrt{I(u^2,u^2)}\sqrt{I(uv,uv)} + 2I(uv,uv) && \text{by (6.11)} \\
&\qquad \geq 0 && \text{by (6.3).}
\end{aligned}$$

As in Section 6.4 Theorem 1.2 now follows for equation (NL) from the results in Sections 3, 4 and 5.

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K	16	$\succ$	12
K_a	16	$u^{\pm}$	28
$K(u)$	32	u_a	16
$\mathcal{K}$	4	V_n	20
$\mathcal{K}^c$	4	W_n	20
$\mathcal{K}_c^d$	4	$X_{a,n}$	18
$\mathcal{K}(c)$	4	X_n	18
κ	28	$Y_{a,n}$	18
κ_n	19	Y_n	18
L	13, 26, 28	Z_n	20

Table 1: List of extra Notation used in Sections 2–5

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