Precise exponential decay for solutions of semilinear elliptic equations and its effect on the structure of the solution set for a real analytic nonlinearity

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We are concerned with the properties of weak solutions of the stationary Schrödinger equation $-\Delta u + Vu = f(u)$, $u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where V is Hölder continuous and $\inf V > 0$. Assuming f to be continuous and bounded near 0 by a power function with exponent larger than 1 we provide precise decay estimates at infinity for solutions in terms of Green's function of the Schrödinger operator. In some cases this improves known theorems on the decay of solutions. If f is also real analytic on $(0, \infty)$ we obtain that the set of positive solutions is locally path connected. For a periodic potential V this implies that the standard variational functional has discrete critical values in the low energy range and that a compact isolated set of positive solutions exists, under additional assumptions.

1 Introduction

We are interested in the properties of weak solutions of

$$(\mathrm{P}) \qquad -\Delta u + Vu = f(u), \qquad u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

where f is continuous, $f(u) \le C|u|^q$ near 0, for some $q > 1$, V is Hölder continuous, bounded, and $\mu_0 := \inf V > 0$.

In the first part of this work we consider exponential decay of solutions of (P). We say that a function u *decays exponentially at infinity with exponent* $\nu > 0$ *if* $\limsup_{|x|\to\infty} \mathrm{e}^{\nu|x|} u(x) < \infty$.

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One of the most thorough studies of this question is an article by Rabier and Stuart [17], where general quasilinear equations are considered. We give a more precise description of the decay of such solutions u in terms of Green's function G of the Schrödinger operator $-\Delta+V$. Setting $H(x) := G(x,0)$ we show that u is bounded above by a multiple of H near infinity. In particular, u decays as fast as H. In some cases this improves the estimates obtained in [17]. To illustrate this, suppose for a moment that V is a positive constant μ_0. Since H decays exponentially at infinity with exponent $\sqrt{\mu_0}$, our result yields the same for every solution of (P), while [17] only yields exponential decay at infinity with exponent ν for every $\nu \in (0, \sqrt{\mu_0})$. Their method could be extended to yield the same result only if $f(u)/u \leq 0$ near 0.

On the other hand, if u is a *positive* solution of (P) then we obtain that u is bounded below by a multiple of H, that is to say, the decay of u and H are *comparable*. We are not aware of a similar result in the literature.

These comparison results are a consequence of *a priori* exponential decay of every solution of (P), of the behavior of f near 0 and of a deep result of Ancona [5] about the comparison of Green's functions for positive Schrödinger operators whose potentials only differ by a function that decays sufficiently fast at infinity.

In the second part of our paper we assume in addition that f is a real analytic function, either on all of $\mathbb{R}$ or solely on $(0,\infty)$. In the complete text analyticity is always *real analyticity*. We have used analyticity before to obtain results on the path connectivity of bifurcation branches and solutions sets [8–10]. Set $F(u) := \int_0^u f$ and introduce the variational functional

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + Vu^2) - \int_{\mathbb{R}^N} F(u)$$

for weakly differentiable functions $u\colon \mathbb{R}^N \to \mathbb{R}$ such that the integrals are well defined. If K is the set of solutions of (P) and K_+ the set of *positive* solutions of (P) then we show that the analyticity of f implies local path connectedness of K in the first case and of K_+ in the second case. Moreover, it follows that J is locally constant on K, respectively K_+. We achieve this by working in spaces of continuous functions with norms weighted at infinity by powers of H. As a consequence, the set K_+ lies in the interior of the positive cone of a related weighted space. This allows to transfer the analyticity from f to the set K_+ in the case where f is only analytic in $(0,\infty)$. From the analyticity of a set its local path connectedness follows from a classical triangulation theorem [12, 14].

In the last part we apply these results to a special case of (P), where we assume V to be periodic in the coordinates. Set $c_0 := \inf J(K) > 0$, the ground state energy. Under additional growth assumptions on f we obtain that $J(K)$, respectively $J(K_+)$, has no accumulation point in the so called *low energy range* $[c_0, 2c_0)$. If in addition V is reflection symmetric and f satisfies an Ambrosetti-Rabinowitz-like condition, an earlier separation Theorem of ours [1] yields, together with the aforementioned conclusion, the existence of a compact set Λ of positive solutions at the ground state energy that is isolated in the set of solutions K.

The latter result is of interest when one considers the existence of so-called *multibump solutions*, which are nonlinear superpositions of translates of solutions in the case of a periodic potential V. It is to be expected that such a set Λ can be used as a base for nonlinear superposition. This would yield a much weaker condition than that imposed in the seminal article [7] and its follow-up works, where the existence of a *single* isolated solution was required.

The present article is structured as follows: In section 2 we study the exact decay of solutions at infinity in terms of Green's function of the Schrödinger operator. Section 3 is devoted to the consequences of analyticity of f. And last but not least, Section 4 treats the consequences for the solution set of (P) if the potential V is periodic.

1.1 Notation

For a metric space (X, d), $r > 0$, and $x \in X$ we denote

$$\begin{aligned} B_r(x; X) &:= \{y \in X \mid d(x, y) < r\}, \\ \overline{B}_r(x; X) &:= \{y \in X \mid d(x, y) \le r\}, \\ S_r(x; X) &:= \{y \in X \mid d(x, y) = r\}. \end{aligned}$$

We also set $B_r X := B_r(0; X)$ if X is a normed space and use analogous notation for the closed ball and the sphere. If X is clear from context we may omit it in the notation. For $k \in \mathbb{N}_0$ denote by $C^k_{\mathrm{b}}(\mathbb{R}^N)$ the space of real valued functions of class C^k on $\mathbb{R}^N$ such that all derivatives up to order k are bounded. We set $C_{\mathrm{b}}(\mathbb{R}^N) := C^0_{\mathrm{b}}(\mathbb{R}^N)$.

2 Exact Decay of Solutions

This section is concerned with comparing the decay of a solution to (P) with Green's function of the Schrödinger operator $T := -\Delta + V$. We show that if the nonlinearity f is well behaved at 0 then a solution decays at least as fast as Green's function. If in addition the solution is positive then it decays at most as fast as Green's function.

Suppose that $N \in \mathbb{N}$. The principal regularity and positivity requirements for the potential we use are contained in the following condition:

(V1) $V \colon \mathbb{R} \to \mathbb{R}$ is Hölder continuous and bounded, and $\mu_0 := \inf V > 0$.

We will need to know *a priori* that weak solutions of (P) and related problems decay exponentially at infinity. For easier reference we include a pertinent result here, even though this fact is in principle well known.

Lemma 2.1. *Assume* (V1)*. Suppose that $f \in C(\mathbb{R})$ satisfies $f(u) = o(u)$ as $u \to 0$ and that $v \in L^\infty(\mathbb{R}^N)$ decays exponentially at infinity. If either $u \in H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a weak solution of $-\Delta u + Vu = f(u)$ or $u \in H^1(\mathbb{R}^N)$ is a weak solution of $-\Delta u + Vu = v$ then u is continuous and decays exponentially at infinity.*

Proof. In the first case we may alter f outside of the range of u in any way we like. Therefore [2, Lemma 5.3] applies and yields, together with standard regularity theory and bootstrap arguments using *a priori* estimates (e.g., [13], Theorem 9.11 and Lemma 9.16), that u is continuous and decays exponentially at infinity.

For the second case suppose that $|v(x)| \le C_1 \mathrm{e}^{-C_2|x|}$ for all $x \in \mathbb{R}^N$, with constants $C_1, C_2 > 0$. For $r > 0$ denote

$$Q(r) := \int_{\mathbb{R}^N \setminus \overline{B}_r} (|\nabla u|^2 + V u^2).$$

We claim that $Q(r)$ decays exponentially at infinity. By contradiction we assume that this were not the case. Then

$$\inf_{r \ge 0} \mathrm{e}^{C_2 r} Q(r) > 0. \tag{2.1}$$

For $r \ge 0$ define the cutoff function ζ_r as in the proof of [2, Lemma 5.3] and set $u_r(x) := \zeta(|x| - r)u(x)$. Let $\delta := \mu_0$. It follows from Hölder's inequality and (2.1) that

$$\begin{aligned}\left| \int_{\mathbb{R}^N} \nabla u \nabla u_r + V u u_r \right| = \left| \int_{\mathbb{R}^N} v u_r \right| &\le C_1 \int_{\mathbb{R}^N \setminus \overline{B}_r} \mathrm{e}^{-C_2|x|} |u(x)| \, \mathrm{d}x \\ &\le C \sqrt{Q(r)} \mathrm{e}^{-C_2 r} \le C Q(r) \mathrm{e}^{-C_2 r/2} \le \frac{\delta}{2} Q(r)\end{aligned}$$

for r large enough. This replaces Equation 5.3 of [2]. As in that proof it follows that

$$\frac{Q(r+1)}{Q(r)} \le \frac{1+\delta}{1+2\delta} < 1$$

for large r, so $Q(r)$ decays exponentially at infinity. Again using standard regularity estimates we obtain that u is continuous and decays exponentially at infinity. □

By [16, Theorem 4.3.3(iii)] the operator T is subcritical, according to the definition in Sect. 4.3 *loc. cit.* Hence T possesses a Green's function $G(x, y)$, i.e., a function that satisfies

$$TG(x, y) = \delta(x - y).$$

Moreover, G is positive. Denote $H(x) := G(x, 0)$ for $x \ne 0$. We collect some properties of H needed later on:

Lemma 2.2. *The function* $H \colon \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ *satisfies:*

(a) $TH \equiv 0$;

(b) $H \in C^2(\mathbb{R}^N \setminus \{0\})$;

(c) $H > 0$;

(d) $\liminf_{x\to 0} H(x) > 0$;

(e) $\limsup_{|x|\to\infty} \mathrm{e}^{\sqrt{\mu_0}\,|x|} H(x) < \infty$.

Proof. (a) and (b) are proved in [16, Theorem 4.2.5(iii)], (c) is a consequence of $G > 0$, and (d) is given by [16, Theorem 4.2.8].

In order to prove (e), consider the function $\psi\colon \mathbb{R}^N \to \mathbb{R}$ given by $\psi(x) := \mathrm{e}^{-\sqrt{\mu_0}\,|x|}$. Then ψ is a supersolution for T on $\mathbb{R}^N\backslash\{0\}$. Take $\alpha > 0$ large enough such that $\alpha\psi \geq H$ on S_1. Denote Green's function for T on B_k with Dirichlet boundary conditions by $\widetilde{G}_k$, for $k \in \mathbb{N}$, and set $\widetilde{H}_k := \widetilde{G}_k(\cdot, 0)$. Then $T\widetilde{H}_k \equiv 0$ on $B_k\backslash\{0\}$ and $\lim_{|x|\to k} \widetilde{H}_k(x) = 0$, by [16, Theorem 7.3.2]. Moreover, [16, Theorem 4.3.7] implies that $\widetilde{H}_k(x) \to H(x)$ as $k \to \infty$, and $(\widetilde{H}_k)$ is an increasing sequence. It follows that $\widetilde{H}_k \leq \alpha\psi$ on S_1 and hence, by the maximum principle, that $\widetilde{H}_k \leq \alpha\psi$ in $\overline{B}_k\backslash B_1$ for all k. Therefore, $H \leq \alpha\psi$ on $\mathbb{R}^N\backslash B_1$ and the claim follows. □

We now state the main result of this section:

Theorem 2.3. (a) *Suppose that* $w \in L^\infty(\mathbb{R}^N)$ *satisfies*

$$|w(x)| \leq C_1 \mathrm{e}^{-C_2|x|}$$

for $x \in \mathbb{R}^N$, *with some fixed* $C_1, C_2 > 0$. *If* $u \in H^1(\mathbb{R}^N)$ *is a weak solution of*

$$-\Delta u + (V - w)u = 0$$

then there exists, for every $\delta > 0$, *some* $R_0 > 0$, *depending only on* δ, N, $\inf V$, $\|V\|_\infty$, C_1 *and* C_2, *such that for every* $R \geq R_0$

$$\limsup_{|x|\to\infty} \frac{|u(x)|}{H(x)} \leq (1+\delta)^2 \max_{x\in S_R} \frac{|u(x)|}{H(x)}. \tag{2.2}$$

In particular,

$$\limsup_{|x|\to\infty} \mathrm{e}^{\sqrt{\mu_0}\,|x|} |u(x)| < \infty. \tag{2.3}$$

(b) *If in addition to the hypotheses of* (a) u *is positive then there exists, for every* $\delta > 0$, *some* $R_0 > 0$, *depending only on* δ, N, $\inf V$, $\|V\|_\infty$, C_1 *and* C_2, *such that for every* $R \geq R_0$

$$\liminf_{|x|\to\infty} \frac{u(x)}{H(x)} \geq (1+\delta)^{-2} \min_{x\in S_R} \frac{u(x)}{H(x)}. \tag{2.4}$$

(c) *If $v \in L^\infty(\mathbb{R}^N)$ satisfies that v/H decays exponentially at ∞ and if $u \in H^1(\mathbb{R}^N)$ is a weak solution of*

$$-\Delta u + Vu = v$$

then there exist continuous functions u_1 and u_2 such that $u = u_1 - u_2$, $Tu_1 = v^+$, $Tu_2 = v^-$, and such that for each $i = 1, 2$ either $u_i \equiv 0$, or $u_i > 0$ and

$$0 < \liminf_{|x|\to\infty} \frac{u_i(x)}{H(x)} \le \limsup_{|x|\to\infty} \frac{u_i(x)}{H(x)} < \infty. \tag{2.5}$$

In particular,

$$\limsup_{|x|\to\infty} \frac{|u(x)|}{H(x)} < \infty.$$

Proof. **(a)** Standard *a priori* estimates, as mentioned in the proof of Lemma 2.1, yield that $u \in L^\infty(\mathbb{R}^N)$. Hence also wu has exponential decay at infinity and Lemma 2.1 yields in particular that

$$u(x) \to 0 \qquad \text{as } |x| \to \infty. \tag{2.6}$$

We take $R > 1$ large enough such that

$$\sup|w| \le \varepsilon_0 := \frac{\mu_0}{2}$$

in $\mathbb{R}^N \backslash \overline{B}_{R-1}$ and define $\eta\colon \mathbb{R}^N \to [0,1]$ by

$$\eta(x) := \begin{cases} 0, & |x| \le R-1, \\ |x| - R + 1, & R-1 \le |x| \le R, \\ 1, & |x| \ge R. \end{cases}$$

Then $\inf(V - \eta w) \ge \varepsilon_0 > 0$. Hence also $T_1 := -\Delta + (V - \eta w)$ is subcritical on $\mathbb{R}^N$ and possesses a positive Green's function G_1. Since we are not assuming w to be locally Hölder continuous, here we refer to [4] and [15] for the existence of the positive Green's function. Set $H_1(x) := G_1(x, 0)$ for $x \ne 0$. In the notation of [5] use our ε_0 and set $r_0 := 1/4$, $c_0 := 1$, and $p := 2N$. Note that the bottom of the spectrum of T and T_1 as operators in L^2 with domain H^2 is greater than or equal to ε_0. Denote

$$\widetilde{C} := \sup\Big\{ \|v\|_{L^N(\overline{B}_{r_0})} \;\Big|\; v \in L^\infty(\overline{B}_{r_0}),\ \|v\|_{L^\infty(\overline{B}_{r_0})} = 1 \Big\}$$

and set $\theta := 1 + \widetilde{C}(C_1 + \|V\|_\infty)$. Define the decreasing function

$$\Psi_R(s) := \begin{cases} C_1 \mathrm{e}^{-C_2(R-1)} & 0 \le s \le R \\ C_1 \mathrm{e}^{-C_2(s-1)} & s \ge R \end{cases}$$

so $\|\eta w\|_{L^\infty(\overline{B}_{r_0}(y))} \le \Psi_R(|y|)$ for $y \in \mathbb{R}^N$. Using these constants, the function Ψ_R and the fact that

$$\lim_{R\to\infty} \int_0^\infty \Psi_R = 0,$$

[5, Theorem 1] yields

$$\frac{1}{1+\delta} H(x) \le H_1(x) \le (1+\delta) H(x) \tag{2.7}$$

for $|x| \ge r_0$ if R is chosen large enough, only depending on δ, N, $\inf(V)$, $\|V\|_\infty$, C_1 and C_2.

The function H_1 is continuous in $\mathbb{R}^N \backslash \{0\}$ and satisfies $T_1 H_1 \equiv 0$ in $\mathbb{R}^N \backslash \{0\}$ in the weak sense. Moreover, $T_1 u \equiv 0$ on $\mathbb{R}^N \backslash \overline{B}_R$ in the weak sense. Set

$$C_3 := (1+\delta)^2 \max_{x \in S_R} \frac{|u(x)|}{H(x)}$$

Then we have by (2.7)

$$|u| \le \frac{C_3}{(1+\delta)^2} H \le \frac{C_3}{1+\delta} H_1 \qquad \text{on } S_R.$$

Note that $H_1(x) \to 0$ as $|x| \to \infty$, by Lemma 2.2(e) and (2.7). Hence (2.6), the maximum principle for weak supersolutions [13, Theorem 8.1] and again (2.7) yield

$$|u| \le \frac{C_3}{1+\delta} H_1 \le C_3 H \qquad \text{on } \mathbb{R}\backslash B_R,$$

that is, (2.2). Together with Lemma 2.2(e) we obtain (2.3).

(b) Define

$$C_4 := (1+\delta)^2 \max_{x \in S_R} \frac{H(x)}{u(x)}.$$

Then (2.7) implies that

$$H_1 \le (1+\delta) H \le \frac{C_4}{1+\delta} u \qquad \text{on } S_R.$$

The maximum principle yields

$$H_1 \le \frac{C_4}{1+\delta} u \qquad \text{on } \mathbb{R}\backslash B_R,$$

so (2.7) implies (2.4).

(c) The operator $T\colon H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ has a bounded inverse by (V1). Denote $v^+ := \max\{0, v\}$ and set $v^- := v^+ - v$. Define $u_1 := T^{-1} v^+ \in H^1(\mathbb{R}^N)$ and $u_2 := T^{-1} v^- \in H^1(\mathbb{R}^N)$. Again we find by Lemma 2.1 that

$$u_i(x) \to 0 \qquad \text{as } |x| \to \infty,\ i = 1, 2.$$

If u_1 is not the zero function then it is positive, by the strong maximum principle. Using

$$\left.\begin{aligned} Tu_1 &\geq 0 \\ TH &= 0 \end{aligned}\right\} \qquad \text{in } \mathbb{R}^N \backslash \{0\}$$

the maximum principle yields

$$0 < \liminf_{|x|\to\infty} \frac{u_1(x)}{H(x)}.$$

Hence also v^+/u_1 decays exponentially at infinity, and (a) implies that

$$\limsup_{|x|\to\infty} \frac{u_1(x)}{H(x)} < \infty.$$

This yields (2.5) for $i = 1$. The case $i = 2$ follows analogously. □

For the semilinear problem (P) we obtain:

Corollary 2.4. *Assume* (V1). *Suppose that* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and that there are* $C, M > 0$ *and* $q > 1$ *such that* $|f(u)| \leq C|u|^q$ *for* $|u| \leq M$. *If* u *is a weak solution of* (P) *then* u *has the properties claimed in Theorem* 2.3, (a) *and* (c). *If in addition* u *is positive, then* u *has the property claimed in Theorem* 2.3(b).

Proof. By our hypotheses on f Lemma 2.1 implies exponential decay of u at infinity. Hence also $w := f(u)/u$ decays exponentially at infinity. Since u is a solution of $-\Delta u+(V-w)u = 0$ Theorem 2.3(a) applies. Therefore also $f(u)/H$ has exponential decay at infinity. These facts yield the claims. □

3 Real Analyticity

Using the precise decay results of the previous section we construct a weighted space Y of continuous functions that contains all solutions of (P) and is such that the positive solutions are contained in the interior of the positive cone of Y. Assuming analyticity of the nonlinearity (on $(0, \infty)$) with appropriate growth bounds we obtain a setting where the (positive) solution set is locally a finite dimensional analytic set and hence locally path connected.

Denote $2^* := \infty$ if $N = 1$ or 2, $2^* := 2N/(N-2)$ if $N \geq 3$ and consider the following conditions on f:

(F1) $f \in C^1(\mathbb{R})$, $f(0) = f'(0) = 0$;

(F2) f is analytic in $\mathbb{R}$ and for every $M > 0$ there are numbers $a_k \in \mathbb{R}$ ($k \in \mathbb{N}_0$) such that

$$\limsup_{k\to\infty} \frac{a_k}{k!} < \infty$$

and

$$|f^{(k)}(u)| \leq a_k |u|^{\max\{0,2-k\}}$$

for $|u| \leq M$ and $k \in \mathbb{N}_0$.

(F3) f is analytic in $\mathbb{R}^+$ and for every $M > 0$ there are numbers $p \in (1, 2^* - 1)$ and $a_k \in \mathbb{R}$ ($k \in \mathbb{N}_0$) such that

$$\limsup_{k \to \infty} \frac{a_k}{k!} < \infty$$

and

$$|f^{(k)}(u)| \leq a_k |u|^{p-k}$$

for $u \in (0, M]$ and $k \in \mathbb{N}_0$; in this case we are only interested in positive solutions of (P) and may take f to be odd, for notational convenience.

(F4) There are $C > 0$ and $\tilde{q} \in (1, 2^* - 1)$ such that $|f(u)| \leq C(1 + |u|^{\tilde{q}})$ for all $u \in \mathbb{R}$.

To give a trivial example of a function satisfying these conditions, take p as in condition (F3). Then $f(u) := |u|^{p-1}u$ satisfies conditions (F1), (F3) and (F4).

If either (F2) or (F3) holds true, then there is $q > 1$ such that for every $M > 0$ there are $a_0, a_1 \in \mathbb{R}$ such that

$$|f(u)| \leq a_0 |u|^q \quad \text{and} \quad |f'(u)| \leq a_1 |u|^{q-1} \qquad \text{if } |u| \leq M. \tag{3.1}$$

To see this take $q := 2$ if (F2) holds true, take $q := p$ if (F3) holds true, and use the respective numbers a_0 and a_1 given for M by these hypotheses.

Denote by K the set of non-zero solutions of (P) and set $K_+ := \{u \in K \mid u \geq 0\}$. Denote by $\mathcal{F}$ the superposition operator induced by f. Then every $u \in K$ satisfies $Tu = \mathcal{F}(u)$. Our goal is to produce a Banach space Y such that

$$\begin{aligned} \Gamma \colon Y &\to Y \\ u &\mapsto u - T^{-1}\mathcal{F}(u) \end{aligned}$$

is well defined and such that $K \subseteq Y$ is the zero set of Γ. Moreover, we need Γ to be a Fredholm map, analytic in a neighborhood of K if (F2) holds true, and analytic in a neighborhood of K_+ if (F3) holds true. In the latter case, because f is not analytic at 0 we need that K_+ belongs to the interior of the positive cone of Y.

Consider the function H defined in Section 2. Pick a number $b_0 \in (0, \infty)$ such that $b_0 \leq \liminf_{x \to 0} H(x)$. By Lemma 2.2 the function $\varphi \colon \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$\varphi(x) := \min\{b_0, H(x)\}$$

is continuous, positive, and has the same decay at infinity as H. Define the spaces

$$X_\alpha := \left\{ u \in C(\mathbb{R}^N) \;\middle|\; \|u\|_{X_\alpha} := \sup_{x \in \mathbb{R}^N} \left| \frac{u(x)}{\varphi(x)^\alpha} \right| < \infty \right\}$$

for $\alpha > 0$. Together with its weighted norm $\|\cdot\|_{X_\alpha}$, X_α is a Banach space. Set

$$Y := X_1 \cap C^1_{\mathrm{b}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$$

and $\|\cdot\|_Y := \|\cdot\|_{X_1} + \|\cdot\|_{C^1_{\mathrm{b}}} + \|\cdot\|_{H^1}$. By (3.1) and Corollary 2.4 $K \subseteq Y$.

We prove the basic properties of the space Y and related mapping properties of the maps T and $\mathcal{F}$:

Lemma 3.1. *Suppose that* (V1), (F1) *and one of* (F2) *or* (F3) *are satisfied. Then the following hold true:*

(a) *$T^{-1}\colon X_\alpha \to Y$ is well defined and continuous if $\alpha > 1$.*

(b) *Let q be given by* (3.1)*. If $\alpha < \min\{2, q\}$ then $\mathcal{F}(Y) \subseteq X_\alpha$, and $\mathcal{F}\colon Y \to X_\alpha$ is completely continuous, i.e., it is continuous and maps bounded sets into relatively compact sets. Moreover, it is continuously differentiable in Y.*

(c) *The set K_+ is contained in the interior of the positive cone of Y.*

(d) *If* (F4) *is satisfied then on K the H^1-topology and the Y-topology coincide.*

Proof. **(a):** For any $s \geq 2$ the linear mapping $T^{-1}\colon L^s(\mathbb{R}^N) \to W^{2,s}(\mathbb{R}^N)$ is well defined and continuous because of (V1). If $v \in X_\alpha \subseteq L^s(\Omega)$ then by the definition of X_α and by Lemma 2.2(e) the function v/H decays exponentially at infinity. For $u := T^{-1}v$ it follows from Theorem 2.3(c) that $u \in X_1$. Therefore

$$\begin{array}{ccc} X_1 & \hookrightarrow & L^s \\ {\scriptstyle T^{-1}}\uparrow & & \uparrow{\scriptstyle T^{-1}} \\ X_\alpha & \hookrightarrow & L^s \end{array}$$

is a commuting diagram of linear maps between Banach spaces, where the inclusions and the map $T^{-1}\colon L^s \to L^s$ are continuous. By the closed graph theorem also $T^{-1}\colon X_\alpha \to X_1$ is continuous. Moreover, if $s > N$ we have continuous maps

$$X_\alpha \hookrightarrow L^s \xrightarrow{T^{-1}} W^{2,s} \hookrightarrow C^1_{\mathrm{b}}$$

so $T^{-1}\colon X_\alpha \to C^1_{\mathrm{b}}$ is continuous. Similarly,

$$X_\alpha \hookrightarrow L^2 \xrightarrow{T^{-1}} H^2 \hookrightarrow H^1$$

and therefore $T^{-1}\colon X_\alpha \to H^1$ is continuous. All in all we have proved (a).

(b): Note that $\mathcal{F}(u) \in X_q \subseteq X_\alpha$ if $u \in X_1$, by (3.1). To see the continuous differentiability of $\mathcal{F}$ in Y, note that f' is locally Hölder (respectively Lipschitz) continuous in $\mathbb{R}$ with exponent $\beta := \min\{1, q-1\}$, as a consequence of (F2) or (F3), respectively. In what follows we repeatedly pick arbitrary $u, v, w \in X_1$ and $C > 0$ such that $|f'(s) - f'(t)| \leq C|s-t|^\beta$ for all $s, t \in \mathbb{R}$ with $|s|, |t| \leq \|u\|_\infty + \|v\|_\infty$. Define $\mathcal{F}_1$ to be the superposition operator induced by f'. First we show that $\mathcal{F}_1(u) \in \mathcal{L}(X_1, X_\alpha)$ as a multiplication operator and that $\mathcal{F}_1 \colon X_1 \to \mathcal{L}(X_1, X_\alpha)$ is continuous. Pick a_1 in (3.1) for $M := \|u\|_\infty$. Then we find

$$\|\mathcal{F}_1(u)w\|_{X_\alpha} \leq a_1 \|\varphi^{q-\alpha}\|_\infty \|u\|_{X_1}^{q-1} \|w\|_{X_1}$$

with $\|\varphi^{q-\alpha}\|_\infty < \infty$ since $\alpha < q$. Hence $\mathcal{F}_1(u) \in \mathcal{L}(X_1, X_\alpha)$. Similarly,

$$\|(\mathcal{F}_1(u) - \mathcal{F}_1(v))w\|_{X_\alpha} \leq C\|\varphi^{\beta+1-\alpha}\|_\infty \|u-v\|_{X_1}^\beta \|w\|_{X_1}$$

with $\|\varphi^{\beta+1-\alpha}\|_\infty < \infty$ since $\alpha < \beta + 1$. Hence

$$\|\mathcal{F}_1(u) - \mathcal{F}_1(v)\|_{\mathcal{L}(X_1, X_\alpha)} \leq C\|\varphi^{\beta+1-\alpha}\|_\infty \|u-v\|_{X_1}^\beta$$

and $\mathcal{F}_1$ is Hölder continuous. For any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}\backslash\{0\}$ there is $\theta_{x,t} \in (-|t|, |t|)$ such that

$$\begin{aligned}\left|\frac{f(u(x)+tv(x)) - f(u(x))}{t} - f'(u(x))v(x)\right| &= |f'(u(x)+\theta_{x,t}v(x)) - f'(u(x))|\,|v(x)| \\ &\leq C|\theta_{x,t}v(x)|^\beta |v(x)| \leq C|v(x)|^{\beta+1}|t|^\beta.\end{aligned}$$

It follows that

$$\left\|\frac{\mathcal{F}(u+tv) - \mathcal{F}(u)}{t} - \mathcal{F}_1(u)v\right\|_{X_\alpha} \leq C\|\varphi^{\beta+1-\alpha}\|_\infty \|v\|_{X_1}^{\beta+1}|t|^\beta$$

and hence that $\mathcal{F}$ is Gâteaux differentiable in u with derivative $\mathcal{F}_1(u)$. Since $\mathcal{F}_1$ is continuous, $\mathcal{F}$ is continuously Fréchet differentiable as a map $X_1 \mapsto X_\alpha$, and thus $Y \hookrightarrow X_1$ implies continuous differentiability of $\mathcal{F} \colon Y \to X_\alpha$.

Suppose now that $(u_n) \subseteq Y$ is bounded in Y and hence bounded in X_1 and $C_{\mathrm{b}}^1(\mathbb{R}^N)$. Passing to a subsequence we can suppose by Arzelà-Ascoli's theorem that (u_n) converges locally uniformly in $\mathbb{R}^N$ to some $u \in C_{\mathrm{b}}(\mathbb{R}^N)$. Since f is uniformly continuous on compact intervals, $\mathcal{F}(u_n)$ converges to $\mathcal{F}(u)$ locally uniformly in $\mathbb{R}^N$. There is $C > 0$ such that

$$u, u_n \leq C\varphi \qquad \text{in } \mathbb{R}^N, \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

For any $\varepsilon > 0$ (3.1) and (3.2) imply that there are a constant $R > 0$, constants $C > 0$, and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds true that

$$\begin{aligned}\frac{|\mathcal{F}(u_n)|}{\varphi^\alpha} &\leq C\varphi^{q-\alpha} \leq \frac{\varepsilon}{3} \qquad \text{in } \mathbb{R}^N\backslash B_R, \\ \frac{|\mathcal{F}(u)|}{\varphi^\alpha} &\leq C\varphi^{q-\alpha} \leq \frac{\varepsilon}{3} \qquad \text{in } \mathbb{R}^N\backslash B_R,\end{aligned}$$

and

$$\frac{|\mathcal{F}(u_n)-\mathcal{F}(u)|}{\varphi^\alpha} \le C\|\mathcal{F}(u_n)-\mathcal{F}(u)\|_\infty \le \frac{\varepsilon}{3} \qquad \text{in } \overline{B}_R.$$

It follows for $n \ge n_0$ that

$$\|\mathcal{F}(u_n)-\mathcal{F}(u)\|_{X_\alpha} \le \sup_{\overline{B}_R} \frac{|\mathcal{F}(u_n)-\mathcal{F}(u)|}{\varphi^\alpha} + \sup_{\mathbb{R}^N\setminus B_R} \frac{|\mathcal{F}(u_n)|+|\mathcal{F}(u)|}{\varphi^\alpha} \le \varepsilon$$

and hence $\mathcal{F}(u_n) \to \mathcal{F}(u)$ in X_α. This proves that $\mathcal{F}$ maps bounded sets in Y into relatively compact sets in X_α. Since $\mathcal{F}$ is differentiable, it is completely continuous.

(c): Fix $u \in K_+$. By Corollary 2.4 there is $C_1 > 0$ such that $C_1\varphi \le u$ in $\mathbb{R}^N$. For any $v \in Y$ such that $\|u-v\|_Y \le C_1/2$ it follows that $\|u-v\|_{X_1} \le C_1/2$ and hence

$$v \ge u - |u-v| \ge \frac{C_1\varphi}{2} > 0$$

in $\mathbb{R}^N$. Therefore, u lies in the interior of the positive cone of Y.

(d): It suffices to prove that on K the H^1-topology is finer than the Y-topology. Therefore, assume that $u_n \to u$ in K with respect to the H^1-topology and suppose by contradiction that $u_n \not\to u$ in Y. Passing to a subsequence we can assume that there is $\delta > 0$ such that

$$\|u_n - u\|_Y \ge \delta \qquad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

By (F4) and standard elliptic regularity estimates, (u_n) is bounded in $C^1_{\mathrm{b}}(\mathbb{R}^N)$. Moreover, the proof of [2, Prop. 5.2] yields, together with regularity estimates, that the functions u_n have a uniform pointwise exponential decay as $|x| \to \infty$. In view of (3.1) we obtain $C_1, C_2 > 0$ such that

$$\frac{f(u_n(x))}{u_n(x)} \le C_1 \mathrm{e}^{-C_2|x|} \qquad \text{for } x \in \mathbb{R}^N,\ n \in \mathbb{N}.$$

By Theorem 2.3(a) (u_n) also remains bounded in X_1 and hence in Y. Pick some $\alpha \in (1, \min\{2, q\})$. By (a) and (b), and passing to a subsequence, $(T^{-1}\mathcal{F}(u_n))$ converges in Y. Since $u_n = T^{-1}\mathcal{F}(u_n)$, $u_n \to v$ in Y, for some $v \in Y$, and $v = u$ since $Y \hookrightarrow H^1$ and $u_n \to u$ in H^1. Hence $u_n \to u$ in Y for this subsequence, contradicting (3.3) and thus finishing the proof of (d). □

If (F1) is satisfied then J, as defined in the introduction, is well defined on Y. The main result of this section is the following

Theorem 3.2. *Assume that* (V1) *and* (F1) *hold true.*

(a) *If* (F2) *is satisfied then K is Y-locally path connected, and J is Y-locally constant on K.*

(b) *If* (F3) *is satisfied then* K_+ *is* Y*-locally path connected, and* J *is* Y*-locally constant on* K_+*.*

Proof. We prove the two statements in parallel. Fix $\alpha \in (1, \min\{2, q\})$, where q is taken from (3.1). For (a) fix $u \in K$, and for (b) fix $u \in K_+$. Set $M := \|u\|_\infty$ and let the numbers a_k be given by (F2) or (F3), respectively.

Denote by $\mathcal{L}^k(X_1, X_\alpha)$ the Banach space of k-linear bounded maps from X_1 into X_α, for $k \in \mathbb{N}_0$ (for $k = 0$ we set $\mathcal{L}^k(X_1, X_\alpha) := X_\alpha$). For $k = 0$ and $k = 1$ we already know that $f^{(k)}(u)$ generates an element of $\mathcal{L}^k(X_1, X_\alpha)$ by multiplication by Lemma 3.1(b). We claim that

(3.4) $f^k(u)$ generates an element A_k of $\mathcal{L}^k(X_1, X_\alpha)$ by multiplication, for every $k \in \mathbb{N}_0$,

$$r_1 := \left(\limsup_{k\to\infty} \|A_k\|^{1/k}_{\mathcal{L}^k(X_1,X_\alpha)} \right)^{-1} > 0, \tag{3.5}$$

and

$$\exists r_2 \in (0, r_1]\ \forall h \in B_{r_2}X_1\ \forall x \in \mathbb{R}^N :\ f(u(x) + h(x)) = \sum_{k=0}^{\infty} \frac{f^{(k)}(u(x))}{k!} h(x)^k. \tag{3.6}$$

To prove the claims in case (a), denote by r_0 the convergence radius of the power series $\sum_0^\infty \frac{a_k}{k!} z^k$. Consider $k \in \mathbb{N}$, $k \geq 2$. Taking into account that $\alpha < 2$ we obtain from (F2) that

$$\left\| \frac{f^{(k)}(u)}{k!} h^k \right\|_{X_\alpha} \leq \frac{a_k}{k!} \sup_{x\in\mathbb{R}^N} \left| \frac{h(x)^k}{\varphi(x)^\alpha} \right| \leq \frac{a_k}{k!} \|\varphi\|_\infty^{k-\alpha} \|h\|_{X_1}^k.$$

Hence (3.4) is true, with

$$\|A_k\|_{\mathcal{L}^k(X_1,X_\alpha)} \leq \frac{a_k}{k!} \|\varphi\|_\infty^{k-\alpha}.$$

Again by (F2), (3.5) is satisfied, and

$$r_2 := \frac{r_0}{\|\varphi\|_\infty} \leq r_1.$$

Suppose now that $h \in B_{r_2}X_1$ and $x \in \mathbb{R}^N$. Then $u(x) \in [-M, M]$ and hence by (F2)

$$\left(\limsup_{k\to\infty} \left| \frac{f^{(k)}(u(x))}{k!} \right| \right)^{-1} \geq r_0.$$

Moreover, $|h(x)| < r_2\varphi(x) \leq r_0$. Since f is analytic, (3.6) follows.

To prove the claims in case (b), denote again by r_0 the convergence radius of the power series $\sum_0^\infty \frac{a_k}{k!} z^k$. By Corollary 2.4 there are $C_1, C_2 > 0$ such that

$$C_1\varphi \leq u \leq C_2\varphi. \tag{3.7}$$

Suppose first that $k \in \mathbb{N}$, $2 \leq k \leq p$. Taking into account that $\alpha < q$ we obtain from (F3)

$$\left\| \frac{f^{(k)}(u)}{k!} h^k \right\|_{X_\alpha} \leq \frac{a_k}{k!} \sup_{x\in\mathbb{R}^N} |u(x)|^{q-k} \left| \frac{h(x)^k}{\varphi(x)^\alpha} \right| \leq \frac{a_k}{k!} \|\varphi^{q-\alpha}\|_\infty C_2^{q-k} \|h\|_{X_1}^k.$$

If $k > q$ then we find

$$\left\| \frac{f^{(k)}(u)}{k!} h^k \right\|_{X_\alpha} \leq \frac{a_k}{k!} \sup_{x\in\mathbb{R}^N} |u(x)|^{q-k} \left| \frac{h(x)^k}{\varphi(x)^\alpha} \right| \leq \frac{a_k}{k!} \|\varphi^{q-\alpha}\|_\infty C_1^{q-k} \|h\|_{X_1}^k.$$

Hence (3.4) is true, with

$$\|A_k\|_{\mathcal{L}^k(X_1,X_\alpha)} \leq \frac{a_k}{k!} \|\varphi^{p-\alpha}\|_\infty C_1^{q-k}$$

for $k > p$. Again by (F3), (3.5) is satisfied, and

$$r_2 := C_1 \min\left\{r_0, \frac{1}{2}\right\} \leq r_1.$$

Suppose now that $h \in B_{r_2} X_1$ and $x \in \mathbb{R}^N$. Then $u(x) \in (0, M]$ and hence by (F3)

$$\left(\limsup_{k\to\infty} \left| \frac{f^{(k)}(u(x))}{k!} \right| \right)^{-1} \geq r_0 u(x).$$

Moreover, $|h(x)| < r_2\varphi(x) \leq C_1 r_0 \varphi(x) \leq r_0 u(x)$, by (3.7), and hence $u(x) + h(x) \geq (C_1 - r_2)\varphi(x) > 0$. Since f is analytic in $(0,\infty)$, (3.6) follows.

For any $h \in X_1$ such that $\|h\|_{X_1} < r_2$ we obtain from (3.5) and $r_2 \leq r_1$ that

$$\sum_{k=0}^{\infty} A_k[h^k] \qquad \text{converges in } X_\alpha. \tag{3.8}$$

Note that X_α embeds continuously in $C_\mathrm{b}(\mathbb{R}^N)$ and that therefore the evaluation E_x at a point $x \in \mathbb{R}^N$ is a bounded linear operator on X_α. Hence for every $x \in \mathbb{R}^N$

$$\begin{aligned}
\mathcal{F}(u+h)(x) &= f(u(x)+h(x)) \\
&= \sum_{k=0}^{\infty} \frac{f^{(k)}(u(x))}{k!} h(x)^k && \text{by (3.6)} \\
&= \sum_{k=0}^{\infty} E_x\left[A_k[h^k]\right] && \text{by (3.4)} \\
&= E_x\left[\sum_{k=0}^{\infty} A_k[h^k]\right] && \text{by (3.8) and } E_x \in \mathcal{L}(X_\alpha,\mathbb{R})
\end{aligned}$$

and therefore

$$\mathcal{F}(u+h) = \sum_{k=0}^{\infty} A_k[h^k], \qquad \text{for all } h \in B_{r_2}X_1. \tag{3.9}$$

By [3, Theorem 6.2] the map $\mathcal{F}$ is analytic in $B_{r_2}X_1$. Since $u \in K_{(+)}$ was arbitrary, $\mathcal{F}\colon X_1 \to X_\alpha$ is analytic in a neighborhood of $K_{(+)}$. And since $Y \hookrightarrow X_1$ and bounded linear operators are analytic, also $\mathcal{F}\colon Y \to X_\alpha$ is analytic in a neighborhood of $K_{(+)}$, c.f. [6, Theorem 7.3].

From the results above we conclude that $\Gamma\colon Y \to Y$ is analytic in a neighborhood of $K_{(+)}$. Moreover, by Lemma 3.1(a) Γ is continuously differentiable in Y, and by Lemma 3.1(b) and [11, Proposition 8.2], for every $v \in Y$ the operator $\mathcal{F}'(v) \in \mathcal{L}(Y, X_\alpha)$ is compact. Hence for every $v \in Y$ the operator $\Gamma'(v)$ is of the form identity minus compact and thus a Fredholm operator of index 0. In short, one calls the map Γ a Fredholm map of index 0.

Recall that $K = \Gamma^{-1}(0)$. By Lemma 3.1(c) K_+ is the set of zeros of Γ in an open neighborhood of u. In any case, the implicit function theorem shows that there are an open neighborhood U of u in Y and a C^∞-manifold $M \subseteq Y$ of finite dimension $\dim \mathcal{N}(\Gamma'(u))$ such that $K \cap U \subseteq M$. In fact, by [6, Theorem 7.5] (see also Corollary 7.3 *loc. cit.*), M is the graph of a analytic map defined on a neighborhood of u in $u + \mathcal{N}(\Gamma'(u))$. Moreover, $K \cap U$ is the set of zeros of the restriction of the finite dimensional analytic map $P\Gamma$ to M. Here $P \in \mathcal{L}(Y)$ denotes the projection with kernel $\mathcal{R}(\Gamma'(u))$ and range $\mathcal{N}(\Gamma'(u))$. Therefore, [14, Theorem 2] applies and yields a triangulation of $K \cap U$ by homeomorphic images of simplexes such that their interior is mapped analytically (see also [12, Satz 4]). This implies that $K_{(+)}$ is locally path connected by piecewise continuously differentiable arcs. Similarly as in the proof of Lemma 3.1 it can be shown that the map $Y \to \mathbb{R}$, $u \mapsto \int F(u)$ is continuously differentiable. Hence also J is continuously differentiable in Y and therefore locally constant on $K_{(+)}$. □

4 Applications to Periodic Potentials

Returning to our main motivation we consider the variational setting in $H^1(\mathbb{R}^N)$. Assuming (V1), (F1), and (F4) the functional J is of class C^1 on $H^1(\mathbb{R}^N)$, and solutions of (P) are in correspondence with critical points of J. Denoting $c_0 := \inf J(K)$ it is easy to see that $c_0 > 0$ if $K \neq \varnothing$.

To inspect the behavior of J on K we will need the following boundedness condition:

(F5) Every sequence $(u_n) \subseteq K$ such that $\limsup_{n\to\infty} J(u_n) < 2c_0$ is bounded.

It is satisfied, for example, under the classical Ambrosetti-Rabinowitz condition. Alternatively, one could use a set of conditions as in [18].

For our purpose we also consider the periodicity condition

(V2) V is 1-periodic in all coordinates.

By concentration compactness arguments c_0 is achieved if (V1), (V2), (F1), (F4) and (F5) hold true and if $K \neq \varnothing$.

The local path connectedness of the set of (positive) solutions of (P) when f is analytic has a consequence on the possible critical levels of J:

Theorem 4.1. *Assume* (V1)*,* (V2)*,* (F1)*,* (F4) *and* (F5)*.*

(a) *If* (F2) *is satisfied, then $J(K)$ has no accumulation point in $[c_0, 2c_0)$.*

(b) *If* (F3) *is satisfied, then $J(K_+)$ has no accumulation point in $[c_0, 2c_0)$.*

Proof. We only prove (a) since the other claim is proved analogously. Assume by contradiction that $J(K)$ contains an accumulation point $c \in [c_0, 2c_0)$. We work entirely in the H^1-topology, which coincides with the Y-topology on K by Lemma 3.1(d). There is a sequence $(u_n) \subseteq K$ such that $J(u_n) \neq c$ and $J(u_n) \to c$. A standard argument using the splitting lemma [2, Proposition 2.5] yields, after passing to a subsequence, a translated sequence $(v_n) \subseteq K$ and $v \in K$ such that $v_n \to v$, $J(v_n) = J(u_n) \neq c$ and $J(v) = c$. Since J is locally constant on K by Theorem 3.2(a) we reach a contradiction. □

We now combine this property with the separation property obtained in [1] to show the existence of compact isolated sets of solutions. For any $c \in \mathbb{R}$ denote

$$K_+^c := \{u \in K_+ \mid J(u) \leq c\}.$$

The result reads:

Corollary 4.2. *In the situation of Theorem* 4.1(b)*, assume in addition that V is of class $C^{1,1}$, that V is even in every coordinate x^i, and that there is $\theta > 2$ such that*

$$f'(u)u^2 \geq (\theta - 1) f(u) u \qquad \textit{for } u \in \mathbb{R}\backslash\{0\}.$$

Suppose that for every $u \in K_+^{c_0}$ that is even in x^i for some $i \in \{1, 2, \ldots, N\}$ it holds true that

$$\int_{\mathbb{R}^N} u^2 \partial_i^2 V \leq 0.$$

(Here we use the weak second derivative of V. It exists because V' is Lipschitz continuous.) Then $K^+ \neq \varnothing$ and there exists a compact subset Λ of $K_+^{c_0}$ that is isolated in K, i.e., that satisfies $\operatorname{dist}(\Lambda, K\backslash\Lambda) > 0$ *in the H^1-metric.*

Proof. By [1, Theorem 1.1] there is a compact subset Λ of $K_+^{c_0}$ such that

$$K_+^{c_0} = \mathbb{Z}^N \star \Lambda \qquad \text{and} \qquad \Lambda \cap \left(\mathbb{Z}^N\backslash\{0\}\right) \star \Lambda = \varnothing.$$

Here $\star$ denotes the action of $\mathbb{Z}^N$ on functions on $\mathbb{R}^N$ by translation: $a \star u := u(\cdot - a)$. It follows easily that

$$\operatorname{dist}(\Lambda, K_+^{c_0}\backslash\Lambda) > 0. \tag{4.1}$$

We claim that $\text{dist}(\Lambda, K\backslash\Lambda) > 0$. Recall that the topologies of the space Y from Section 3 and the H^1-topology coincide on K and that K_+ is contained in the interior of the positive cone of Y, by Lemma 3.1(d) and (c). Hence $\text{dist}(\Lambda, K\backslash K_+) > 0$. It remains to show that $\text{dist}(\Lambda, K_+\backslash\Lambda) > 0$. Assume by contradiction that this were not the case. Since Λ is compact there would exist a sequence $(u_n) \subseteq K_+\backslash\Lambda$ and $u \in \Lambda$ such that $u_n \to u$. Since c_0 is not an accumulation point of $J(K_+)$ by Theorem 4.1(b), $(u_n) \subseteq K_+^{c_0}$. But this contradicts (4.1), proving the claim. □

Note that in [1] we show how to construct concrete examples that satisfy the conditions of Corollary 4.2.

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