# An invariant set generated by the domain topology for parabolic semiflows with small diffusion

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**Abstract**

We consider the singularly perturbed semilinear parabolic problem $u_t - d^2\Delta u + u = f(u)$ with homogeneous Neumann boundary conditions on a smoothly bounded domain $\Omega \subseteq \mathbb{R}^N$. Here $f$ is superlinear at 0 and $\pm\infty$ and has subcritical growth. For small $d > 0$ we construct a compact connected invariant set $X_d$ in the boundary of the domain of attraction of the asymptotically stable equilibrium 0. The main features of $X_d$ are that it consists of positive functions that are pairwise non-comparable, and that its topology is at least as rich as the topology of $\partial\Omega$ in a certain sense. If the number of equilibria in $X_d$ is finite this implies the existence of connecting orbits within $X_d$ that are not a consequence of a well known result by Matano.

## 1. Introduction

For $N \geq 2$, a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary, a small positive parameter $d$ and a continuously differentiable map $f\colon \mathbb{R} \to \mathbb{R}$ we consider the dynamics of the parabolic boundary value problem

$$(\mathrm{P}_d) \qquad \begin{cases} u_t - d^2\Delta u + u = f(u) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\partial_\nu u$ denotes the derivative of $u$ with respect to the outer normal of $\partial\Omega$, $u_t$ denotes the time derivative, and $\Delta u$ the $x$-Laplacian of $u$, as usual. We assume that $f$ has

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superlinear but subcritical growth at 0 and $\pm\infty$. This problem is well posed for initial data in $E := H^1(\Omega)$ and induces a (local) continuous semiflow $\varphi_d$ on $E$. We are only interested in nonnegative solutions and set $E^+ := \{u \in E \mid u \geq 0 \; a.e.\}$. As a consequence of the parabolic maximum principle $E^+$ is positive invariant under $\varphi_d$.

The corresponding stationary problem

$$(\mathrm{E}_d) \qquad \begin{cases} -d^2\Delta u + u = f(u) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases}$$

has been the object of study for many authors. Usually it is treated as a variational problem. Denoting $F(u) := \int_0^u f(s)\,\mathrm{d}s$, the variational functional is defined on $E$ by

$$J_d(u) := \frac{1}{2}\int_\Omega (d^2|\nabla u|^2 + u^2)\,\mathrm{d}x - \int_\Omega F(u)\,\mathrm{d}x\ .$$

It is well known that $J_d$ presents a strict Lyapunov function for $\varphi_d$. Since $f(u) = o(u)$ as $u \to 0$, 0 is an asymptotically stable equilibrium of $\varphi_d$. The domain of attraction of 0,

$$\mathcal{A}_d := \{\, u \in E \mid \varphi_d^t(u) \to 0 \text{ as } t \to \infty \,\},$$

is an open neighborhood of 0 and its boundary $\partial\mathcal{A}_d$ is a closed subset of $E$. Clearly $\mathcal{A}_d$, $\mathcal{A}_d^+ := \mathcal{A}_d \cap E^+$, $\partial\mathcal{A}_d$ and $\partial\mathcal{A}_d^+ := \partial\mathcal{A}_d \cap E^+$ are positive invariant. It was first shown by Poláčik [20] that the boundary $\partial\mathcal{A}_d^+$ is a Lipschitz submanifold of codimension 1. Assuming condition (F5) below Lazzo and Schmidt [13] proved that a semiorbit starting at $u \in E^+ \setminus \overline{\mathcal{A}_d^+}$ blows up in finite time. Hence $\partial\mathcal{A}_d^+$ separates the blow up solutions in $E^+$ from those converging to 0. This blow up phenomenon has been widely studied in recent years; see e.g. [7,8] and the references therein. The present paper is a contribution to the dynamics on the separatrix $\partial\mathcal{A}_d^+$. General results about the flow in $\overline{\mathcal{A}_d}$ which hold for arbitrary $d > 0$ can be found in our recent papers [2, 3].

We denote by

$$\mathcal{K}_d := \{\, u \in E \mid J_d'(u) = 0,\ u > 0 \,\}$$

the set of positive equilibria of $\varphi_d$. Under the hypotheses stated below $\mathcal{K}_d$ is not empty, $\mathcal{K}_d \subseteq \partial\mathcal{A}_d^+$, and $J_d$ achieves a positive minimum $a_d$ on $\partial\mathcal{A}_d^+$, necessarily at an element of $\mathcal{K}_d$. Note that $\lim_{d\to 0} a_d d^{-N} > 0$ is well defined (cf. [1, 19]). Fixing some small $\epsilon_0 > 0$ which will be determined later, and letting $d \to 0$ we focus on the dynamics in $\partial\mathcal{A}_d^+$ with energy below

$$(1.1) \qquad c_d := a_d + \epsilon_0 d^N.$$

We write $J_d^c := \{u \in E \mid J_d(u) \leq c\}$ for the sublevel sets as usual and call the equilibria in $\mathcal{K}_d \cap J_d^{c_d}$ *low-energy equilibria*.

In order to state our result, we set $p_S = (N+2)/(N-2)$ for $N \geq 3$ and $p_S = \infty$ for $N = 2$, and we assume the following hypotheses:

(F1) $f \in C^1(\mathbb{R}, \mathbb{R})$, (F2) $f(0) = f'(0) = 0$,

(F3) there are $p \in (1, p_S)$ and $a_1 > 0$ such that $|f'(u)| \le a_1(|u|^{p-1} + 1)$ for all $u \in \mathbb{R}$.

As a consequence of these conditions, ($P_d$) defines a continuous semiflow $\varphi_d$ on $E = H^1(\Omega)$ as described above. We are interested in the flow in $\overline{\mathcal{A}_d^+}$. Therefore, values of $f$ in $(-\infty, 0)$ can be prescribed at will, and we may assume $f$ to be odd. We require three additional hypotheses on $f$:

(F4) There are $\theta > 2$ and $a_2 \ge 0$ such that $f(u)u \ge \theta F(u) - a_2$ for all $u > 0$.

(F5) Either $f'(u)u > f(u)$ for all $u > 0$, or there is $\mu > 1$ such that $f'(u)u \ge \mu f(u)$ for all $u > 0$, and $f(u)$ is positive if $u > 0$ is large enough.

(F6) For fixed $d > 0$ the following hold: Every semiorbit starting in $\overline{\mathcal{A}_d^+}$ exists for all time. If $A$ is a relatively compact subset of $\overline{\mathcal{A}_d^+}$ then $\bigcup_{t\ge 0} \varphi_d^t(A)$ is relatively compact.

Note that the first alternative in (F5) implies that $f(u)/u$ is strictly increasing in $u > 0$, hence $f(u) > 0$ for all $u > 0$. The second alternative in (F5) allows for $f(u)$ to be negative for a bounded range of $u$. Finally, as in [2] the compactness of $\varphi_d$ and [21, Theorem 3.1] imply that (F6) follows from (F1)–(F5) if we assume the existence of $a_3, a_4 > 0$ and $r \in (0, p]$ such that $r$ is close to $p$ and

$$f(u) \ge a_3 u^r - a_4 \tag{1.2}$$

for $u > 0$.

As a standard example, suppose that $n \in \mathbb{N}$, $1 < p_1 < p_2 < \cdots < p_n < p_S$, and $b_k \in \mathbb{R}$ ($k = 1, 2, \dots, n$). If for some $k_0 \in \{1, 2, \dots, n\}$ it holds that $b_k < 0$ for $k < k_0$ and $b_k > 0$ for $k \ge k_0$, then (F1)–(F6) apply to $f$ given by

$$f(u) = \sum_{k=1}^{n} b_k u^{p_k} \qquad \text{for } u \ge 0.$$

In particular, (F3) is satisfied with $p = p_n$, (F4) is satisfied with $\theta = p_{k_0} + 1$, the second alternative of (F5) is satisfied with $\mu = p_{k_0}$, and (1.2) is satisfied with $r = p = p_n$. As noted above, together with (F1) and (F2) this implies (F6).

The topology of $\partial\Omega$ plays an important rôle for the number and location of low energy equilibria. In order to describe this we denote the barycenter of $u \in L^2(\Omega) \setminus \{0\}$ with respect to the $L^2$-norm $|\,.\,|_2$ by

$$\beta(u) := \frac{1}{|u|_2^2} \int_\Omega |u(x)|^2 x \, \mathrm{d}x \, .$$

Given $r > 0$ then $\beta(u) \in U_r(\partial\Omega)$ for any $u \in \mathcal{K}_d \cap J_d^{c_d}$ if $d$ is small enough, and $\#(\mathcal{K}_d \cap J_d^{c_d}) \ge \operatorname{cat} \partial\Omega$. Here cat denotes the Lusternik-Schnirelmann category of a topological space. Similar and more refined results can be found in [1] where the barycenter with respect to the $H^1$-norm was considered. Our main theorem shows that the dynamics of the parabolic semiflow ($P_d$) is also strongly influenced by the topology of $\partial\Omega$.

**Theorem 1.1.** *Assume* (F1) - (F5) *hold. Let* $C$ *be a connected component of* $\partial\Omega$ *and fix* $r > 0$*. Then there is* $d_0 > 0$ *such that for* $d \in (0, d_0)$ *there exists a set* $X_d \subset \partial\mathcal{A}_d^+ \cap J_d^{c_d}$ *with the following properties:*

(i) $X_d$ *is compact, connected and invariant under* $\varphi_d$*. The restriction of* $\varphi_d$ *to* $X_d$ *is a global flow and* $X_d$ *consists entirely of positive functions in* $C(\overline{\Omega})$*.*

(ii) $\beta(u) \in U_r(C) := \{\, x \in \mathbb{R}^N \mid \operatorname{dist}(x, C) < r \,\}$ *for* $u \in X_d$*.*

(iii) $\operatorname{cat}(X_d) \geq \operatorname{cat}(C)$*, where* $\operatorname{cat}(X_d)$ *is defined via open coverings.*

(iv) $H^*(C)$ *is a direct summand of* $H^*(X_d)$*, where* $H^*$ *denotes Alexander-Spanier or Čech cohomology with any coefficients.*

(v) $\dim X_d \geq N - 1$*, where* dim *denotes covering dimension.*

(vi) $X_d$ *contains at least* $k := \operatorname{cat}(X_d)$ *equilibria. If* $X_d$ *contains only finitely many equilibria then it contains* $k$ *equilibria* $u_1, \ldots, u_k$ *and connecting orbits from* $u_{j+1}$ *to* $u_j$*,* $j = 1, \ldots, k-1$*.*

**Remarks 1.2.** (a) The statements Theorem 1.1(iii), (iv) can be interpreted as saying that $X_d$ is topologically at least as complicated as $C$. Motivated by the relation between critical points of the mean curvature function $H\colon \partial\Omega \to \mathbb{R}$ and the barycenter and maximum points of low energy equilibria (cf. [5, 11, 15, 19, 24]) we expect that the parabolic flow in $X_d$ is closely related to the flow on $C \subset \partial\Omega$ generated by $\nabla H$. In fact, generically we expect that $X_d$ is a manifold diffeomorphic to $C$ and that the two flows are flow equivalent.

(b) Throughout this paper cat denotes the Lusternik-Schnirelmann category defined via open coverings (see Section 3). This is not essential for manifolds or absolute neighborhood retracts (ANRs) but it does make a difference here. In particular, $X_d$ may not be an ANR in general.

(c) Imposing homogeneous Dirichlet boundary conditions it is known that $\operatorname{cat}(\Omega)$ is a lower bound for the number of positive low energy equilibria. Replacing $\partial\Omega$ with $\Omega$ we think that a result similar to Theorem 1.1 holds, with the exception of Theorem 1.1(v).

(d) In our setting no two positive equilibria are comparable. Hence the existence of connecting orbits does not follow from the results in Matano [17].

(e) We need to work with Alexander-Spanier or Čech cohomology because we need the continuity property which is not satisfied by singular cohomology. We refer the reader to the books [6] for Čech and [22] for Alexander-Spanier cohomology.

(f) Under certain nondegeneracy and smoothness conditions on $f$ it is shown in Henry [12, Theorem p. 105] that for fixed $d$, generically with respect to domain variation, all equilibria of $\varphi_d$ are hyperbolic. In this case the set $X_d$ contains only finitely many equilibria. We thank the referee for drawing our attention to this reference.

**Example 1.3.** Suppose $\Omega \subset \mathbb{R}^N$ has the shape of a solid torus with boundary $\partial\Omega$ homeomorphic to $(S^1)^{N-1}$. Then $\partial\Omega$ is connected and $\mathrm{cat}(\partial\Omega) = N$. Thus we find for $d$ small a compact invariant set $X_d \subset \partial\mathcal{A}_d^+$ with $\mathrm{cat}(X_d) \geq N$ and $\dim X_d \geq N-1$. It consists only of low energy equilibria and connecting orbits between these. If it contains only finitely many equilibria then it contains a chain of $N$ equilibria and connecting orbits as stated in Theorem 1.1.

## 2. A retraction up to homotopy

We state some basic properties of the solutions of $(\mathrm{P}_d)$. For every $u_0 \in E$ there is a solution $u(t)$ of $(\mathrm{P}_d)$ with initial data $u_0$, defined for times $t$ in a maximal interval $[0,T)$ with $T \in (0,\infty]$. It can be viewed as an element of

$$C([0,T),E) \cap C((0,T),H^2(\Omega)) \cap C^1((0,T),L^2(\Omega)) \ .$$

We denote the associated continuous semiflow by $\varphi_d$ and write $\varphi_d^t(u_0) = u(t)$. The energy $J_d$ satisfies

$$\frac{d}{dt}J_d(u(t)) = -|\dot{u}(t)|_2^2$$

for $t > 0$. Therefore $J_d$ decreases strictly along nonconstant flow lines, and there is a one-to-one correspondence between equilibria of $\varphi_d$ and critical points of $J_d$. Assumption (F2) implies that 0 is a linearly, hence asymptotically stable equilibrium of $\varphi_d$. Assumption (F6) implies that every semiorbit starting in $\overline{\mathcal{A}_d}$ has an $\omega$-limit set that is nonempty, connected, and consists entirely of equilibria.

Recall that we denote by $\mathcal{K}_d$ the set of positive equilibria of $\varphi_d$. Using (F5) it is easy to see that every equilibrium in $E \setminus \{0\}$ is linearly unstable. It follows from the results in [13] (see also [9,20]) that $\mathcal{K}_d \subseteq \partial\mathcal{A}_d^+$. These facts and the results of [16,18] imply that $J_d$ achieves a positive minimum $a_d$ on $\partial\mathcal{A}_d^+$ at an element of $\mathcal{K}_d$. In [13] it is also observed that $(\mathrm{P}_d)$ exhibits a threshold phenomenon in the following sense: For $u \in E^+ \setminus \{0\}$ there is a threshold value

$$\alpha(u) := \sup\{\, s \geq 0 \mid su \in \mathcal{A}_d \,\} \tag{2.1}$$

with the properties

- $0 < \alpha(u) < \infty$
- $su \in \mathcal{A}_d$ for $s \in [0, \alpha(u))$
- $\alpha(u)u \in \partial\mathcal{A}_d$
- the solution of $(\mathrm{P}_d)$ with initial value $su$ blows up in finite time if $s > \alpha(u)$.

These results were proved for homogeneous Dirichlet boundary conditions, but the same arguments apply in our setting.

The technique to derive asymptotic estimates for ($E_d$) as $d \to 0$ has been developed by many authors. Our main reference will be the paper [1]. In a standard way it can be shown that the results of [1] are valid in our setting even though the assumptions on $f$ used there are slightly stronger than ours.

A key rôle will be played by positive solutions of the elliptic problem on the whole space:

$$(E_\infty) \qquad -\Delta u + u = f(u)\,, \quad u \in H^1(\mathbb{R}^N)\,.$$

With respect to the corresponding variational functional

$$I_\infty(u) := \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\,\mathrm{d}x\,,$$

a *ground state solution* is by definition a positive solution of ($E_\infty$) that minimizes $I_\infty$ among all positive solutions. It is well known that under our conditions such minimizers exist and that they are radially symmetric about some point in $\mathbb{R}^N$ and decrease exponentially at infinity, cf. [4, 10, 14]. We do not know whether or not a ground state is unique up to translations. The energy level of a ground state solution will be denoted by $b_\infty$ in the sequel.

Since the minimum $a_d$ of $J_d$ on $\partial\mathcal{A}_d$ coincides with the minimum of $J_d$ on $\mathcal{K}_d$, [1, Prop. 3.4] yields that

$$(2.2) \qquad a_d = d^N(b_\infty/2 + o(1)) \quad \text{as } d \to 0.$$

Next we define a continuous map from $\partial\Omega$ into sublevel sets of the restriction of $J_d$ to $\partial\mathcal{A}_d$. Fix a ground state solution $w$ of ($E_\infty$) that is radially symmetric about 0. Define a map $\kappa_d\colon \partial\Omega \to E$ by setting

$$\kappa_d(P)(x) = w\left(\frac{|x-P|}{d}\right)$$

for $P \in \partial\Omega$ and $x \in \Omega$. Clearly $\kappa_d$ is continuous, and it follows from [1, Prop. 3.2] (there $\kappa_d$ was defined using cut-off functions; the proof clearly extends to our setting) that

$$(2.3) \qquad \max_{t\geq 0} J_d(t\kappa_d(P)) = d^N(b_\infty/2 + o(1))$$

as $d \to 0$, uniformly in $P \in \partial\Omega$.

Recall the definition of the threshold value $\alpha(u)$ for $u > 0$ given in (2.1). In view of $\alpha(u)u \in \partial\mathcal{A}_d$ we define

$$\gamma_d\colon \partial\Omega \to \partial\mathcal{A}_d^+\,, \qquad \gamma_d(P) := \alpha(\kappa_d(P))\kappa_d(P)\,.$$

It is clear that

$$(2.4) \qquad \operatorname{dist}(P, \beta(\gamma_d(P))) \to 0 \qquad \text{as } d \to 0, \text{ uniformly in } P.$$

**Lemma 2.1.** *The map $\gamma_d$ is continuous. It holds that $J_d(\gamma_d(P)) = d^N(b_\infty/2 + o(1))$ as $d \to 0$, uniformly in $P$.*

*Proof.* As usual, for $u, v \in H^1(\Omega)$ we write $u > v$ if $u \geq v$ and $u \neq v$, and we write $u \gg v$ if $u, v \in C(\overline{\Omega})$ and if $u - v$ is an element of the interior of the positive cone in $C(\overline{\Omega})$. Recall that $\varphi_d$ is strongly order preserving: If $u > v$ then $\varphi_d^t(u) \gg \varphi_d^t(v)$ for all $t > 0$ where the orbits exist. Also recall that $\varphi^t$ is a continuous map from its domain of definition into $C(\overline{\Omega})$ for $t > 0$.

For continuity it suffices to prove that $\alpha$ is continuous. Let us consider a sequence $(u_n) \subseteq E^+ \setminus \{0\}$ with $u_n \to u \neq 0$ as $n \to \infty$. We may assume that $\alpha(u_n) \to \alpha^* \in (0, \infty]$ as $n \to \infty$ since $\mathcal{A}_d$ is a neighborhood of 0.

If $\alpha^* > \alpha(u)$ then there is $\bar{\alpha} > \alpha(u)$ such that $\alpha(u_n) \geq \bar{\alpha}$ for large $n$. By the remarks made above, $\varphi_d^1(\bar{\alpha}u) \gg \varphi_d^1(\alpha(u)u)$ and hence by continuity $\varphi_d^1(\alpha(u_n)u_n) \geq \varphi_d^1(\bar{\alpha}u_n) \gg \varphi_d^1(\alpha(u)u)$ for large $n$. As $\varphi_d^1(\alpha(u_n)u_n) \in \partial\mathcal{A}_d \cap E^+$ we get $\alpha(u)u \in \mathcal{A}_d$, a contradiction. If $\alpha^* < \alpha(u)$ then $\alpha^* u \in \mathcal{A}_d$. As $\mathcal{A}_d$ is open also $\alpha(u_n)u_n \in \mathcal{A}_d$ for $n$ large enough, a contradiction. Accordingly, $\alpha(u) = \alpha^*$ which proves continuity of $\alpha$.

The asymptotic estimate follows from (2.2) and (2.3). □

**Proposition 2.2.** *For every $r > 0$ there exist $\varepsilon_0 > 0$ such that*

$$\operatorname{dist}(\beta(u), \partial\Omega) < r \quad \text{for } u \in \partial\mathcal{A}_d \cap J_d^{c_d}$$

*for small $d$; here $c_d = a_d + \varepsilon_0 d^N$ as in (1.1).*

*Proof.* The proof will make use of a technique commonly used when dealing with singularly perturbed problems, namely *blow-up analysis.* For this purpose we introduce for $d > 0$ the scaled domain

$$\Omega_d := \{\, x \in \mathbb{R}^N \mid dx \in \Omega \,\}$$

and consider the scaled problem

$$(\mathrm{SP}_d) \qquad \begin{cases} v_t - \Delta v + v = f(v) & \text{in } \Omega_d, \\ \partial_\nu v = 0 & \text{on } \partial\Omega_d. \end{cases}$$

Solutions $u$ of $(\mathrm{P}_d)$ and $v$ of $(\mathrm{SP}_d)$ are in a one-to-one correspondence via scaling of the $x$-variable: $v(t, x) = u(t, dx)$. Consequently, the dynamic properties of $(\mathrm{P}_d)$ carry over to $(\mathrm{SP}_d)$ via scaling. We denote the parabolic semiflow which $(\mathrm{SP}_d)$ induces on $E_d := H^1(\Omega_d)$ by $\psi_d$. The domain of attraction of 0 with respect to $\psi_d$ will be denoted by $\mathcal{B}_d$. The variational functional for the scaled stationary equation

$$\begin{cases} -\Delta v + v = f(v) & \text{in } \Omega_d, \\ \partial_\nu v = 0 & \text{on } \partial\Omega_d \end{cases}$$

is given by

$$I_d(v) := \frac{1}{2} \int_{\Omega_d} (|\nabla v|^2 + v^2) \, \mathrm{d}x - \int_{\Omega_d} F(v) \, \mathrm{d}x \, ,$$

and its minimum on $\partial\mathcal{B}_d$ by $b_d$. Note that for $u \in E$, and $v$ in $E_d$ its scaled counterpart

$$J_d(u) = d^N I_d(v)$$

and hence

$$b_d = b_\infty/2 + o(1) \tag{2.5}$$

as $d \to 0$ by (2.2).

Arguing by contradiction we assume that there are $r > 0$, a sequence $d_n \to 0$ and elements $u_n \in \partial\mathcal{A}_{d_n} \cap J_{d_n}^{a_{d_n}+\frac{1}{n}d_n^N}$ such that

$$\operatorname{dist}(\beta(u_n), \partial\Omega) \geq r\ .$$

Going forward in time a small amount for each $n$, replacing the value of $r$ with $r/2$, and using that the semiflow $\varphi_{d_n}$ and the map $\beta$ are continuous on $E$, we may as well assume that the solution of ($\mathrm{SP}_{d_n}$) starting at $u_n$ can be considered as a map in $C([0,\infty), H^2(\Omega)) \cap C^1([0,\infty), L^2(\Omega))$ (note that we gain differentiability into $L^2(\Omega)$ up to the initial time 0 by this).

Denoting the $L^2$-barycenter of $v \in L^2(\Omega_d) \setminus \{0\}$ by

$$\beta_d(v) := \frac{1}{|v|_2^2} \int_{\Omega_d} |v(x)|^2 x \,\mathrm{d}x$$

and rescaling we obtain elements

$$v_n \in \partial\mathcal{B}_{d_n} \cap I_{d_n}^{b_{d_n}+1/n} \tag{2.6}$$

with

$$\operatorname{dist}(\beta_{d_n}(v_n), \partial\Omega_{d_n}) \geq \frac{r}{d_n}\ . \tag{2.7}$$

The strategy to obtain a contradiction is to produce elements $\overline{v}_n \in E_{d_n}$ that are close to $v_n$ in $L^2(\Omega_{d_n})$ such that $\operatorname{dist}(\beta_{d_n}(\overline{v}_n), \partial\Omega_{d_n})$ remains bounded as $n \to \infty$.

Let us fix $n \in \mathbb{N}$ for the moment. Consider the closure $M_n$ of the trajectory starting at $v_n$, i.e.

$$M_n := \overline{\{\, \psi_{d_n}(t, v_n) \mid t \geq 0 \,\}} \quad \text{in } E_{d_n}.$$

Since $\partial B_{d_n}$ is closed and positive invariant, $M_n$ is a subset of $\partial B_{d_n}$ and hence $I_{d_n}$ is bounded below on $M_n$ by $b_{d_n}$. Moreover, assumption (F6) implies that $M_n$ is compact. Since $E_{d_n}$ is continuously embedded in $L^2(\Omega_{d_n})$ (which is a Hausdorff space) the topologies of $E_{d_n}$ and $L^2(\Omega_{d_n})$ coincide on $M_n$. Therefore $I_{d_n}$ is continuous on $M_n$ with respect to the $L^2(\Omega_{d_n})$-topology. From now on we denote the $L^2$-norm of $v \in L^2(\Omega_{d_n})$ by $|v|_2$.

The variational principle of Ekeland, cf. [23, Thm. I.5.1], yields $\overline{v}_n \in M_n$ such that

$$|v_n - \overline{v}_n|_2 \leq \frac{1}{\sqrt{n}} \tag{2.8}$$

$$I_{d_n}(\overline{v}_n) \leq I_{d_n}(v_n) \tag{2.9}$$

$$I_{d_n}(\overline{v}_n) \leq I_{d_n}(w) + \frac{1}{\sqrt{n}}|w - \overline{v}_n|_2 \quad \text{for all } w \in M_n. \tag{2.10}$$

We claim that

$$\|DI_{d_n}(\overline{v}_n)\|_{\mathcal{L}(E_{d_n},\mathbb{R})} \le \frac{1}{\sqrt{n}}\,. \tag{2.11}$$

If $\overline{v}_n \in \omega(v_n)$ (the $\omega$-limit set of $v_n$) then $\overline{v}_n$ is an equilibrium point of $\psi_{d_n}$. Hence $\overline{v}_n$ is a critical point of $I_{d_n}$ and (2.11) holds. If $\overline{v}_n \notin \omega(v_n)$ then there is $t_n \ge 0$ such that $\overline{v}_n = \psi_{d_n}(t_n, v_n)$. Recall that we have set up things such that the map given by $t \mapsto \psi_{d_n}(t, v_n)$ is in $C^1([0,\infty), L^2(\Omega_{d_n}))$. Denote by $w_n(t)$ the solution of $(\mathrm{SP}_{d_n})$ with initial datum $\overline{v}_n$, i.e. $w_n(t) := \psi_{d_n}(t, \overline{v}_n)$. Note that

$$\frac{d}{dt} I_{d_n}(w_n(t)) = -|\dot{w}_n(t)|_2^2\,, \tag{2.12}$$

where $\dot{w}_n$ denotes the derivative of the $C^1$-map $w_n\colon [0,\infty) \to L^2(\Omega_{d_n})$. We obtain from (2.10) for $t \ge 0$ that

$$0 \le I_{d_n}(w_n(0)) - I_{d_n}(w_n(t)) \le \frac{1}{\sqrt{n}} |w_n(0) - w_n(t)|_2$$

and hence from (2.12) that

$$|\dot{w}_n(0)|_2 = \lim_{t \to 0+} \frac{|I_{d_n}(w_n(0)) - I_{d_n}(w_n(t))|}{|w_n(0) - w_n(t)|_2} \le \frac{1}{\sqrt{n}}\,.$$

Now (2.11) follows since $w_n(0) \in H^2(\Omega)$, $w_n$ satisfies equation $(\mathrm{SP}_{d_n})$ for $t \ge 0$, and hence

$$\|DI_{d_n}(w_n(0))\|_{\mathcal{L}(E_{d_n},\mathbb{R})} \le |\dot{w}_n(0)|_2$$

holds.

Combining (2.5), (2.6), (2.9), (2.11), [1, Lemma 2.10], and [1, Prop. 3.6] yields that $\|\overline{v}_n\|_{E_{d_n}}$ remains bounded as $n \to \infty$, and that there is a ground state solution $w$ of $(\mathrm{E}_\infty)$ and a sequence $(y_n) \subseteq \mathbb{R}^N$ with $y_n \in \partial\Omega_{d_n}$ such that

$$\|\overline{v}_n - w(\cdot - y_n)|_{\Omega_{d_n}}\|_{E_{d_n}} \to 0 \tag{2.13}$$

and such that $\overline{v}_n$ is concentrated in $y_n$ in the following sense:

$$\forall \epsilon > 0\ \exists R_0 > 0\ \forall R > R_0\ \forall n \in \mathbb{N}\colon \int_{\Omega_{d_n} \setminus B_R(y_n)} (|\nabla \overline{v}_n|^2 + |\overline{v}_n|^2)\,\mathrm{d}x \le \epsilon. \tag{2.14}$$

As the boundaries of the scaled sets $\Omega_d$ behave uniformly with respect to $d \in (0,1]$ there exists $C > 0$ such that $|w(\cdot - y_n)|_2 \ge C$, and therefore by (2.13)

$$|\overline{v}_n|_2 \ge \frac{C}{2} \quad \text{for } n \text{ large enough.} \tag{2.15}$$

Consider $\rho > 0$ such that $\Omega \subseteq B_\rho(0)$. Fix $\epsilon > 0$ and define for each $n$ a probability measure $\mu_n$ on $\mathbb{R}^N$ by setting

$$\mu_n(A) := \frac{1}{|v_n|_2^2} \int_A |v_n|^2\,\mathrm{d}x$$

for every Lebesgue measurable subset $A$ of $\mathbb{R}^N$. From (2.14) and (2.15) it follows that there is $R > 0$ with

$$\mu_n(\Omega_{d_n} \setminus B_R(y_n)) \le \epsilon$$

for all $n \in \mathbb{N}$. Along the lines of the proof of [1, Prop. 3.5] it follows easily that

$$\operatorname{dist}(\beta_{d_n}(\overline{v}_n), \partial\Omega_{d_n}) \le 2\frac{\epsilon\rho}{d_n} + R\,.$$

On the other hand, (2.8) and the boundedness of $\|\overline{v}_n\|_{E_{d_n}}$ imply boundedness of $|\overline{v}_n|_2$ and $|v_n|_2$. Therefore (2.15) yields $C > 0$ with

$$\operatorname{dist}(\beta_{d_n}(v_n), \beta_{d_n}(\overline{v}_n)) \le \frac{C\rho}{d_n}|v_n - \overline{v}_n|_2 \le \frac{C\rho}{d_n}\frac{1}{\sqrt{n}}\,.$$

Choosing $\epsilon$ small and $n$ large enough we reach a contradiction with (2.7). □

**Remark 2.3.** In contrast to the results in [1] we cannot prove Proposition 2.2 when defining the barycenter $\beta(u)$ via the $H^1$-norm.

Fix $\delta > 0$ such that

$$\Gamma := \{\, x \in \mathbb{R}^N \mid \operatorname{dist}(x, \partial\Omega) < \delta \,\} \tag{2.16}$$

is a normal tubular neighborhood of $\partial\Omega$. Denote by $\pi\colon \Gamma \to \partial\Omega$ the corresponding normal projection. Now we obtain the main result of this section.

**Corollary 2.4.** *For every $r \in (0, \delta]$ there are $\epsilon_0 > 0$ and $d_0 > 0$ such that for all $d \in (0, d_0]$ the maps*

$$\partial\Omega \xrightarrow{\gamma_d} \partial\mathcal{A}_d^+ \cap J_d^{c_d} \xrightarrow{\beta} U_r(\partial\Omega) \xrightarrow{\pi} \partial\Omega \tag{2.17}$$

*are well defined and such that $\pi \circ \beta \circ \gamma_d$ is homotopic to the identity on $\partial\Omega$. In other words, $\partial\Omega$ is a homotopy retract of $\partial\mathcal{A}_d^+ \cap J_d^{c_d}$.*

*Proof.* Choose $\epsilon_0$ as in Proposition 2.2. Together with Lemma 2.1 this implies that $\beta(\partial\mathcal{A}_d^+ \cap J_d^{c_d}) \subseteq U_r(\partial\Omega)$ and that $\gamma_d(\partial\Omega) \subseteq \partial\mathcal{A}_d^+ \cap J_d^{c_d}$ for small $d$. Using (2.4) fix $d_0$ small enough such that for $d \in (0, d_0]$ in addition to these properties it also holds that $\beta(\gamma_d(P)) \in B_{2\delta/3}(P) \subseteq \Gamma$ for all $P \in \partial\Omega$. Hence the segment with the endpoints $P$ and $\beta(\gamma_d(P))$ is included in $\Gamma$ for all $P \in \partial\Omega$. The linear homotopy $h$ from the inclusion $\partial\Omega \to \mathbb{R}^N$ to the map $\beta \circ \gamma_d$ has its image in $\Gamma$, and $\pi \circ h$ defines a homotopy from $\mathrm{id}_{\partial\Omega}$ to $\pi \circ \beta \circ \gamma_d$. □

This result has strong consequences for the topology of and the dynamics in $\partial\mathcal{A}_d^+ \cap J_d^{c_d}$ as we will see below.

## 3. Proof of Theorem 1.1

Recall that we are given a connected component $C$ of $\partial\Omega$ and $r > 0$. We may assume that $r \le \delta$, where $\delta$ is given in the definition of $\Gamma$ in (2.16). We choose $\epsilon_0 > 0$ and $d_0 > 0$ as in Corollary 2.4 and fix $d \in (0, d_0]$. The maps $\gamma_d$, $\beta$ and $\pi$ induce restrictions

$$C \xrightarrow{\gamma_d} W_d := (\pi \circ \beta)^{-1}(C) \xrightarrow{\beta} U_r(C) \xrightarrow{\pi} C$$

such that

$$\pi \circ \beta \circ \gamma_d \text{ is homotopic to } \mathrm{id}_C. \tag{3.1}$$

Observe that $W_d$ is a closed subset of $\partial\mathcal{A}_d^+ \cap J_d^{c_d}$ and that it is positive invariant under $\varphi_d$ because $C$ is a connected component of $\partial\Omega$.

Let $X_d := \omega(\gamma_d(C))$ be the $\omega$-limit set of $\gamma_d(C)$ in $W_d$:

$$X_d = \{u \in E \mid \varphi_d^{t_n}(\gamma_d(x_n)) \xrightarrow{n\to\infty} u \text{ for some } x_n \in C,\ t_n \to \infty\}.$$

Being the $\omega$-limit set of the connected set $\gamma_d(C)$, $X_d$ is connected and $\varphi_d$ is a global flow on $X_d$. By (F6) $X_d$ is compact. Standard regularity theory and the strong comparison principle imply that $X_d$ consists of functions that are continuous and positive in $\overline{\Omega}$. It is also clear that $\beta(u) \in U_r(C)$ for $u \in X_d$. Hence we have proved (i) and (ii) of Theorem 1.1.

For the proof of (iii) and (vi) we need the Lusternik-Schnirelmann category $\mathrm{cat}_Z(A)$ where $Z$ is a topological space and $A \subset Z$. This is the smallest integer $k \ge 0$ such that there exist open sets $U_1, \dots, U_k \subset Z$ with $A \subset U_1 \cup \ldots \cup U_k$ and which are contractible in $Z$, that is there exists a continuous map $h_i \colon U_i \times [0,1] \to Z$ with $h_i(z,0) = z$ and $h_i(z,1) = z_i \in Z$ for all $z \in U_i$, $i = 1, \dots, k$. If such a covering does not exist then $\mathrm{cat}_Z(A) := \infty$. Note that $\mathrm{cat}_Z(A) = 0$ if and only if $A = \varnothing$. We also write $\mathrm{cat}(Z) := \mathrm{cat}_Z(Z)$, as usual. It is important here that we work with open coverings and not with closed ones as it is often the case. The two definitions are equivalent if $Z$ is an ANR. However, we shall apply the results to $Z = X_d$ and we do not know whether $X_d$ is an ANR. The following properties are standard and easy to prove.

(c1) $A \subset B \subset Z \quad \Rightarrow \quad \mathrm{cat}_Z(A) \le \mathrm{cat}_Z(B)$.

(c2) For any $A \subset Z$ there exists a neighborhood $V$ of $A$ in $Z$ with $\mathrm{cat}_Z(V) = \mathrm{cat}_Z(A)$.

(c3) $A, B \subset C \quad \Rightarrow \quad \mathrm{cat}_Z(A \cup B) \le \mathrm{cat}_Z(A) + \mathrm{cat}_Z(B)$.

(c4) Given $V \subset Z$ open, $h \colon V \times [0,1] \to Z$ continuous with $h_0(z) = z$ we have $\mathrm{cat}_Z(A) \le \mathrm{cat}_Z(h_1(A))$ for every $A \subset V$; here $h_t = h(\cdot, t)$.

In fact, (c1) and (c3) are trivial. (c2) is also trivial because we work with open coverings: If $A \subset U_1 \cup \ldots \cup U_k$ is a covering as in the definition of $\mathrm{cat}_Z(A)$ then set $V := U_1 \cup \ldots \cup U_k$. Finally, in order to see (c4) let $h_1(A) \subset U_1 \cup \ldots \cup U_k$ be a covering as in the definition of $\mathrm{cat}_Z(h_1(A))$. Then $A \subset h_1^{-1}(U_1) \cup \ldots \cup h_1^{-1}(U_k)$ is an open covering of $A$, and each $h_1^{-1}(U_i)$ can first be deformed into $U_i$ using $h$, then into a point since $U_i$ is contractible in $Z$.

*Proof of Theorem* 1.1(iii). By (c2) there exists a neighborhood $V$ of $X_d$ in $W_d$ with $\mathrm{cat}_{W_d}(X_d) = \mathrm{cat}_{W_d}(V)$. Since $\gamma_d(C)$ is compact there exists $T > 0$ with $\varphi^T(\gamma_d(C)) \subset V$, hence $\mathrm{cat}_{W_d}(V) \geq \mathrm{cat}_{W_d}(\gamma_d(C))$ by (c4). It remains to prove $\mathrm{cat}_{W_d}(\gamma_d(C)) \geq \mathrm{cat}(C)$. In order to see this, let $h\colon C\times[0,1] \to C$ be a homotopy between $h_0 = \mathrm{id}_C$ and $h_1 = \pi\circ\beta\circ\gamma_d$. Let $\gamma_d(C) \subset U_1 \cup \ldots \cup U_k$ be a covering as in the definition of $\mathrm{cat}_{W_d}(\gamma_d(C))$. Setting $V_j := \gamma_d^{-1}(U_j)$ defines an open covering $C = V_1 \cup \ldots \cup V_k$ of $C$. It remains to show that each $V_j$ is contractible in $C$. There exists a homotopy $h^{(j)}\colon U_j \times [0,1] \to W_d$ which deforms $U_j$ to a point. Then

$$V_j \times [0,1] \to C, \quad (x,t) \mapsto \begin{cases} h(x,2t) & 0 \leq t \leq 1/2 \\ \pi\beta(h^{(j)}(\gamma_d(x),2t-1)) & 1/2 \leq t \leq 1 \end{cases}$$

deforms $V_j$ to a point. □

*Proof of Theorem* 1.1(iv). This is a consequence of the continuity property [22, Theorem 6.6.2] of Alexander-Spanier cohomology which we recall here for the reader's convenience. Given topological spaces $A \subset Z$ and $\xi \in H^*(Z)$ we set $\xi|_A := i^*(\xi)$ where $i\colon A \hookrightarrow Z$ denotes the inclusion and $i^*\colon H^*(Z) \to H^*(A)$ the induced homomorphism in cohomology. Now the continuity property says that for a paracompact Hausdorff space $Z$ and a closed subset $A$, given $\xi \in H^*(A)$ there exists a neighborhood $V$ of $A$ in $Z$ and $\eta \in H^*(V)$ with $\eta|_A = \xi$. If $V_1$, $V_2$ are two such neighborhoods and $\eta_1 \in H^*(V_1)$, $\eta_2 \in H^*(V_2)$ satisfy $\eta_1|_A = \eta_2|_A = \xi$ then there exists a neighborhood $V_3 \subset V_1 \cap V_2$ of $A$ so that $\eta_1|_{V_3} = \eta_2|_{V_3}$.

For the proof of Theorem 1.1(iv) we construct a homomorphism $\sigma\colon H^*(X_d) \to H^*(C)$ such that $\sigma \circ (\pi \circ \beta)^* = \mathrm{id}$ on $H^*(C)$. Then $H^*(X_d) \cong H^*(C) \oplus \mathrm{kern}(\sigma)$. Given $\xi \in H^*(X_d)$ there exists a neighborhood $V$ of $X_d$ in $W_d$ and $\eta \in H^*(V)$ with $\eta|_{X_d} = \xi$. There also exists $T > 0$ such that $\varphi_d^t(\gamma_d(C)) \subset V$ for all $t \geq T$. Then we set $\sigma(\xi) := (\varphi_d^t \circ \gamma_d)^*(\eta)$, any $t \geq T$. This is independent of $t \geq T$ because the maps $\varphi_d^{t_1} \circ \gamma_d$, $\varphi_d^{t_2} \circ \gamma_d\colon C \to V$ are homotopic. The definition is also independent of $V$ and $\eta$. If $V_1, V_2$ are neighborhoods of $X_d$ in $W_d$, and $\eta_1 \in H^*(V_1)$, $\eta_2 \in H^*(V_2)$ satisfy $\eta_1|_{X_d} = \xi = \eta_2|_{X_d}$ then there exists a neighborhood $V_3 \subset V_1 \cap V_2$ of $X_d$ with $\eta_1|_{V_3} = \eta_2|_{V_3}$. For $t$ large we have $\varphi_d^t(\gamma_d(C)) \subset V_3$. Therefore

$$(\varphi_d^t \circ \gamma_d)^*(\eta_1) = (\varphi_d^t \circ \gamma_d)^*(\eta_1|_{V_3}) = (\varphi_d^t \circ \gamma_d)^*(\eta_2|_{V_3}) = (\varphi_d^t \circ \gamma_d)^*(\eta_2).$$

Here we interpret $\varphi_d^t \circ \gamma_d$ as a map $\varphi_d^t \circ \gamma_d\colon C \to V_j$ for $j = 1, 2, 3$.

We have seen that $\sigma\colon H^*(X_d) \to H^*(C)$ is well defined. In order to see that $\sigma \circ (\pi \circ \beta)^* = \mathrm{id}$ consider $\zeta \in H^*(C)$ and set $\xi := (\pi \circ \beta|_{X_d})^*(\zeta)$. Let $V$ be a neighborhood of $X_d$ in $W_d$, $\eta \in H^*(V)$ with $\eta|_{X_d} = \xi$, so that $\sigma(\xi) = (\varphi_d^t \circ \gamma_d)^*(\eta)$ for $t$ large. Then $(\pi \circ \beta|_V)^*(\zeta)|_{X_d} = \xi = \eta|_{X_d}$, hence by the continuity property of $H^*$ there exists a neighborhood $V_1 \subset V$ of $X_d$ with $(\pi \circ \beta|_V)^*(\zeta)|_{V_1} = \eta|_{V_1}$. For $t$ large we have $(\varphi_d^t \circ \gamma_d)(C) \subset V_1$, so that

$$\sigma(\xi) = (\varphi_d^t \circ \gamma_d)^*(\eta|_{V_1}) = (\varphi_d^t \circ \gamma_d)^*((\pi \circ \beta|_V)^*(\zeta)|_{V_1}) = (\pi \circ \beta \circ \varphi_d^t \circ \gamma_d)^*(\zeta) = \zeta.$$

The last equality follows from the fact that $\pi \circ \beta \circ \varphi_d^t \circ \gamma_d\colon C \to C$ is homotopic to $\pi \circ \beta \circ \gamma_d$ which is homotopic to $\mathrm{id}_C$ by (3.1). □

*Proof of Theorem* 1.1(v). Since $C$ is a compact $(N-1)$-dimensional manifold without boundary we have $H^{N-1}(C) \neq 0$. Now Theorem 1.1(iv) implies $H^{N-1}(X_d) \neq 0$ which is only possible if $\dim X_d \geq N-1$. □

*Proof of Theorem* 1.1(vi). This is a consequence of Theorem 1.1(iii) and Theorem 4.1 below. □

## 4. Existence of connecting orbits

We state a rather general result concerning the existence of connecting orbits for gradient-like flows. Let $X$ be a compact metric space with metric $d$. Let $\varphi$ be a gradient-like flow on $X$ with strict Lyapunov function $f\colon X \to \mathbb{R}$, i. e. $f$ is continuous and $f(\varphi^t(x)) < f(x)$ for $x \in X$, $t > 0$, except when $x$ is a stationary solution. The set of stationary solutions is denoted by $S$, and we assume that it is finite. Hence also the set $f(S)$ of "critical values" is finite. As a consequence, the $\alpha$- and $\omega$-limit sets of $x \in X$ consist of a single equilibrium which we denote by $\alpha(x), \omega(x) \in S$.

**Theorem 4.1.** *If $X$ is connected then there exist $k := \mathrm{cat}(X) = \mathrm{cat}_X(X)$ equilibria $x_1, \dots, x_k$ and connecting orbits from $x_{j+1}$ to $x_j$, $j = 1, \dots, k-1$.*

The proof requires some preparations. For $r > 0$ and $x \in X$ denote by $B_r(x)$ the closed ball with radius $r$ and center $x$. We fix $r > 0$ such that $B_r(x_0) \cap S = \{x_0\}$ and $f(B_r(x_0)) \cap f(S) = \{f(x_0)\}$ for all $x_0 \in S$. Moreover, for $x_0 \in S$ let

$$W^{\mathrm{u}}(x_0) := \{x \in X \mid \varphi^t(x) \to x_0 \text{ as } t \to -\infty\}$$

denote the unstable set of $x_0$ and define

$$S_r W^{\mathrm{u}}(x_0) := \{x \in W^{\mathrm{u}}(x_0) \mid\ d(x, x_0) = r\},$$
$$B_r W^{\mathrm{u}}(x_0) := \{x \in W^{\mathrm{u}}(x_0) \mid d(x, x_0) \leq r\}.$$

**Lemma 4.2.** *Suppose that $x_0 \in S$. If $(y_n) \subseteq \partial B_r(x_0)$ and if there exist $s_n > 0$ such that $\varphi^{-s_n}(y_n) \to x_0$ as $n \to \infty$ then $y_n \to y \in S_r W^{\mathrm{u}}(x_0)$ along a subsequence.*

*Proof.* For large $n$ there exist $t_n \in [0, s_n)$ with $\varphi^{-t_n}(y_n) \in \partial B_r(x_0)$ and $\varphi^{-t-t_n}(y_n) \in B_r(x_0)$ for all $t \in [0, s_n - t_n]$. We may assume that $\varphi^{-t_n}(y_n) \to z \in \partial B_r(x_0)$ as $n \to \infty$ since $X$ is compact. Next, $s_n - t_n \to \infty$ because otherwise $s_n - t_n \to t$ along a subsequence, hence $\varphi^{-s_n}(y_n) = \varphi^{-s_n+t_n}(\varphi^{-t_n}(y_n)) \to \varphi^{-t}(z) \neq x_0$, a contradiction. For $t \geq 0$ it holds that $\varphi^{-t}(z) = \lim_{n\to\infty} \varphi^{-t-t_n}(y_n) \in B_r(x_0)$. Since $x_0$ is the only equilibrium in $B_r(x_0)$ it follows that $\lim_{t\to\infty} \varphi^{-t}(z) = x_0$ and $z \in S_r W^{\mathrm{u}}(x_0)$. Hence by our choice of $r$ there exists $T > 0$ with $f(\varphi^T(z)) < \min f(B_r(x_0))$, and therefore $f(\varphi^T(\varphi^{-t_n}(y_n)) < \min f(B_r(x_0))$ for $n$ large. Since $f(\varphi^{t_n}(\varphi^{-t_n}(y_n))) = f(y_n) \geq \min f(B_r(x_0))$ it follows that $t_n \leq T$ for $n$ large. Thus we may assume $t_n \to t$ as $n \to \infty$. This implies

$$y_n = \varphi^{t_n}(\varphi^{-t_n}(y_n)) \to y := \varphi^t(z) \in W^{\mathrm{u}}(x_0).$$

□

For convenience we introduce some notation: For $x, y \in S$ we write $x \succ y$ if there exists a connecting orbit from $x$ to $y$. A sequence $x = x_0 \succ x_1 \succ \ldots \succ x_j = y$ of equilibria is called a heteroclinic chain from $x$ to $y$ of length $j$.

**Lemma 4.3.** *For $x, y \in S$ with $x \neq y$ and $y \in \overline{W^{\mathrm{u}}(x)}$ there exists a heteroclinic chain from $x$ to $y$.*

*Proof.* There exist sequences $y_n \in S_r W^{\mathrm{u}}(x)$, $t_n > 0$, such that $\varphi^{t_n}(y_n) \to y$. By Lemma 4.2 we may assume that $y_n \to z_1 \in S_r W^{\mathrm{u}}(x)$ and set $x_1 := \omega(z_1)$. Clearly $f(x_1) < f(x)$ and hence $x_1 \neq x$.

If $x_1 = y$ we are done. If $x_1 \neq y$ there exist sequences $s_n < r_n$ with $\varphi^{s_n}(y_n) \to x_1$ and $\varphi^{r_n}(y_n) \in \partial B_r(x_1)$. As a consequence of Lemma 4.2 $\varphi^{r_n}(y_n) \to z_2 \in S_r W^{\mathrm{u}}(x_1)$ along a subsequence and we set $x_2 := \omega(z_2)$. Now $f(x_2) < f(x_1)$ and hence $x_2 \notin \{ x_1, x \}$. We have heteroclinic orbits $x \succ x_1 \succ x_2$. As above, either $x_2 = y$ and we are done, or $x_2 \neq y$ and we continue as before. After a finite number of steps we arrive at a heteroclinic chain from $x$ to $y$. □

**Lemma 4.4.** *If $X$ is connected then $X$ is path-connected.*

*Proof.* For each $x \in X$ there exists a path to $\omega(x) \in S$. Consequently, $X$ can have at most finitely many path-components. It follows from Lemma 4.3 that $\overline{W^{\mathrm{u}}(x)}$ is path connected for every $x \in S$. This implies that a path component $Y$ of $X$ can be written as $Y = \bigcup_{x \in Y \cap S} \overline{W^{\mathrm{u}}(x)}$, hence it is closed. Since $X$ has only finitely many path-components, each is closed and open. But then $X$ being connected can have only one path-component. □

*Proof of Theorem* 4.1. We define the height $h(x) \in \mathbb{N}_0$ of an equilibrium $x \in S$ by

$$h(x) := \max\{ j \in \mathbb{N}_0 \mid \text{ there exists a heteroclinic chain of length } j \text{ starting at } x \}.$$

The height of a stable equilibrium is 0. Theorem 4.1 can be formulated as saying that there exists a heteroclinic chain of length $k-1$ in $X$, provided there are only finitely many equilibria. For $j \in \mathbb{N}_0$ we consider the set

$$X^j := \{ x \in X \mid h(\alpha(x)) \leq j \}.$$

We have to prove that $X^{k-1} \setminus X^{k-2} \neq \varnothing$. By Lemma 4.3 we know:

$$x \in S,\ h(x) \leq j \qquad \Rightarrow \qquad \overline{W^{\mathrm{u}}(x)} \subset X^j.$$

We claim that $\mathrm{cat}_X(X^j) \leq j+1$. For $j = 0$, $X^0$ consists precisely of the stable equilibria. So $X^0$ is finite, hence $\mathrm{cat}_X(X^0) = 1$ because $X$ is path-connected by Lemma 4.4. Moreover, $X^{j+1} = X^j \cup (X^{j+1} \setminus X^j)$ and therefore $\mathrm{cat}_X(X^{j+1}) \leq \mathrm{cat}_X(X^j) + \mathrm{cat}_X(X^{j+1} \setminus X^j)$. For $x \in X^{j+1} \setminus X^j$ we have $h(\alpha(x)) = j+1$. Clearly there cannot exist a connecting orbit between equilibria having the same height. Using the flow $\varphi^t$ for $t \to -\infty$, the set $X^{j+1} \setminus X^j$ can be deformed to the set $S^{j+1} = \{ x \in S \mid h(x) = j+1 \}$ which is finite. Using the properties of the category $\mathrm{cat}_X$ we obtain $\mathrm{cat}_X(X^{j+1} \setminus X^j) \leq 1$, hence $\mathrm{cat}_X(X^{j+1}) \leq \mathrm{cat}_X(X^j) + 1$.

Since $\mathrm{cat}(X) = k$ we deduce $X^{k-2} \neq X$, hence $X^{k-1} \setminus X^{k-2} \neq \varnothing$. □

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